Chapter 2.3

Theory of Fluctuations in Plasma

C.R. Oberman and E. A. Williams

Plasma Physics Laboratory
Princeton University
Princeton, New Jersey 08544, USA

Laboratory for Laser Energetics
University of Rochester
Rochester, New York 14627, USA

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Handbook of Plasma Physics, Eds. M.N. Rosenbluth and R.Z. Sagdeev
Volume 1: Basic Plasma Physics I, edited by A.A. Galeev and R.N. Sudan

2.3.1. Introduction

Motivation

This article will be concerned with the theory of fluctuations in plasma. In the previous article on the kinetic theory of waves at the Vlasov level of description, an initial small amplitude excitation or wave in the plasma (in the absence of an external driver) would simply die away (Landau or collisional damping) if the plasma is stable, or grow to some level determined by some nonlinear interactions and/or by somehow altering the background free-energy source responsible for the growth.

These nonlinear mechanisms will be elaborated upon in the following articles in this series.

However, even in stable plasma, particle discreteness can constantly re-excite these fluctuations (Cerenkov emission by single particles, longitudinal and transverse bremsstrahlung from collisions of pairs of particles, etc.) so that even if the mean electric field $\langle \boldsymbol{E}(\boldsymbol{x}, t) \rangle = 0$, for example, in thermal equilibrium, the expectation, or mean, $\langle E^2(x, t) \rangle$, e.g., will not be zero. (The terminology and notation will be elaborated upon immediately in the following sections.) The presence of these fluctuations in general are indeed responsible for diffusion and transport in the plasma.

Even in globally stable inhomogeneous plasmas, if the local dispersion relation $\varepsilon[\boldsymbol{k}(\boldsymbol{x}), \omega]$ signals only weak stability or instability then these fluctuations can rise to locally high levels and the concomitant transport may be very large.

The scattering of electromagnetic radiation from electron-density fluctuations has proven a powerful diagnostic in determining the structure of plasma. For example, laser-scattering is uniformly used to determine the electron temperature and density in many laboratory devices. As another example, scattering from density fluctuations in the ionosphere has led to remarkable confirmation of the theory of plasma behavior, and if the launched wave is sufficiently intense [but still not strong enough to induce parametric instabilities (electron–ion decay)] another emission process, secondary induced scattering, can dominate the determination of the fluctuation level, and again predictions of fluctuation theory have served remarkably well.

The theory of fluctuations in a plasma was inspired by the possibility of measuring ionospheric plasma density by radar backscatter suggested by Gordon (1958). The necessary theory was given in a series of papers by Salpeter (1960), Dougherty and Farley (1960), Fejer (1960) and Rostoker (1961). This latter paper [see also Dupree (1963) and Klimontovich and Silin (1962)] laid out the systematic foundations of the theory of fluctuations. Much of what will be presented in this article are more recent simplifications, clarifications, and extensions of that work (Williams, 1973; Krommes and Oberman, 1976).

This chapter remains within the confines of perturbation theory, low-order expansion in powers of the plasma parameter, $\varepsilon \equiv 1/n\lambda_D^3$, or $E^2/8\pi nT$ (here $T \equiv k_B T$, is the temperature in energy units). Since generally only linearly stable plasma will be considered, this perturbation theory is usually but not always (!) valid (see Chapters 4.1 and 4.4). Possible breakdown or indeed failure of the formulation, and the need for renormalization, etc., will only be pointed at.

Scope of work

In Section 2.3.2, first the one- and two-time hierarchies will be developed and then it will be shown how expectations are computed; next the irreducible cluster expansions are introduced and then related to the Klimontovich equation for the fluctuating microdensity.

Section 2.3.3 concerns itself with the theory of fluctuations to lowest order in ε for the most part in homogeneous plasma. The relation between power spectra and correlation functions is shown and the Kramers-Kronig relations are developed; furthermore, the simplicity of the formulation to compute the power spectrum for charge density and electric field fluctuations is shown, both out of and in thermal equilibrium where the fluctuation-dissipation theorem is illustrated. In the next subsection the kinetic equation for the one-particle distribution function for a multi-species plasma is developed. How to include a magnetic field is pointed out in the next subsection, and in the following subsection the expression for scattering of small-amplitude electromagnetic waves from electron density fluctuations is derived. Then the induced emission when the pump intensity is higher is considered, and the scattering mechanism shifts from that due to electrons and the electronic component of the polarization clouds around both electrons and ions, to scattering up and down shifted from the moving ions. Here the Superposition Principle of Rostoker (1964) [see also Krommes (1976)] is stated and employed. Some remarks on inhomogeneous plasma will also be given.

Section 2.3.4 argues for a description of fluctuations on the kinetic time scale, and finally Section 2.3.5 demonstrates a theory of hydrodynamic fluctuations.

2.3.2. One- and two-time hierarchies, expectations, cluster expansions and relation to Klimontovich formalism

The hierarchy equations for correlation functions

Consider an s-species fully ionized plasma contained in a volume V in which there are N_r particles of the rth species with charge e_r and mass m_r. Suppose that the plasma is neutral overall, that is

$$\sum_{r=1}^{s} N_r e_r = 0. \tag{1}$$

Assume also that the plasma is nonrelativistic and hence that the interparticle interactions are adequately described as electrostatic. The state of the plasma is thus completely described by a point in Γ-space, a $6N$ dimensional space ($N = \Sigma_r N_r$) with one dimension for each position and each velocity coordinate of each particle in the plasma. The entire history of the plasma is contained in the trajectory of this point in Γ-space, in principle obtainable by solving the Newtonian N-body problem. Such a solution even if available, would contain a vast excess of detailed information associated with the particular initial conditions of the problem. The procedure introduced by Gibbs is to envisage an ensemble of realizations of the plasma and attempt only to describe the evolution of ensemble averaged quantities.

We shall consider an ensemble of plasma described by the ensemble density $D_1(Y,t)$, where Y is a point in Γ-space. $D_1(Y,t)\,\mathrm{d}Y$ is the probability that at time t there is a member of the ensemble in the volume $\mathrm{d}Y$ around Y. Similarly, we define a two-time ensemble density $D_2(Y_0,t_0;Y,t)$ such that $D_2(Y_0,t_0;Y,t)\,\mathrm{d}Y_0\,\mathrm{d}Y$ is the joint probability that a member of the ensemble is in $(Y_0, Y_0+\mathrm{d}Y_0)$ at time t_0 and is in $(Y, Y+\mathrm{d}Y)$ at time t. Both D_1 and D_2 satisfy the Liouville equation in the Y,t variables.

$$\left[\frac{\partial}{\partial t} + \sum_{n=1}^{N}\left(\boldsymbol{v}_n\cdot\frac{\partial}{\partial \boldsymbol{x}_n} + \frac{e_n}{m_n c}\boldsymbol{v}_n\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial \boldsymbol{v}_n}\right)\right.$$
$$\left. - \sum_{\substack{l,n=1\\ n\neq l}}^{N} \frac{\partial}{\partial \boldsymbol{x}_n}\frac{e_l e_n}{m_n|\boldsymbol{x}_n-\boldsymbol{x}_l|}\cdot\frac{\partial}{\partial \boldsymbol{v}_n}\right]\left\{\begin{matrix} D_1(Y,t) \\ D_2(Y_0,t_0;Y,t)\end{matrix}\right\} = 0, \tag{2}$$

where $Y=(X_1,\ldots X_n)$ and $X_i=(\boldsymbol{x}_i,\boldsymbol{v}_i)$. $\boldsymbol{B}_0$ is an external magnetic field. D_2 has a singular initial condition,

$$D_2(Y_0,t_0;Y,t_0) = D_1(Y_0,t_0)\,\delta(Y-Y_0)., \tag{3}$$

whereas D_1 is assumed to have a smooth initial condition.

D_1 and D_2 are normalized so that

$$\int \mathrm{d}Y D_1(Y,t) = 1, \qquad \iint \mathrm{d}Y\,\mathrm{d}Y_0 D_2(Y_0,t_0;Y,t) = 1. \tag{4}$$

The preservation of these normalizations in time is guaranteed by Liouville's theorem.

It may be supposed without loss of generality that D_1 and D_2 are symmetric under interchanges of like particles. Since this is clearly also a symmetry of the Hamiltonian, it is likewise preserved in time.

The following definitions are made:

$$f_s^{r_1\cdots r_s}(X_1, X_2\cdots X_s, t) \equiv V^s\int D_1(Y,t)\,\mathrm{d}X_{s+1}\,\mathrm{d}X_{s+2}\cdots\mathrm{d}X_N, \tag{5}$$

upon integrating $D_1(Y,t)$ over all except s particles of species $r_1, r_2, \ldots r_s$.

$$F_s^{r_1;r_2\cdots r_{s+1}}(X_1,t;X_2'\cdots X_{s+1}',t')$$
$$\equiv V^{s+1}\int D_2(Y,t;Y',t')\,\mathrm{d}X_2\cdots\mathrm{d}X_N\,\mathrm{d}X_1'\,\mathrm{d}X_{s+2}'\cdots\mathrm{d}X_N', \tag{6}$$

upon integrating D_2 over all the unprimed particles except one of the species r_1, and over all except s of the primed particles of species $r_2, r_3, \ldots r_{s+1}$. The s primed particles are distinct from the singled out unprimed one.

$$\Omega_s^{r_1;r_1\ldots r_s}(X_1,t;X_1'\cdots X_s',t')$$
$$\equiv \frac{V^{s+1}}{N_{r_1}}\int D_2(Y,t;Y',t')\,\mathrm{d}X_2\cdots\mathrm{d}X_N\,\mathrm{d}X_{s+1}'\cdots\mathrm{d}X_N', \tag{7}$$

upon integrating D_2 over all unprimed particles except one of species r_1, and over all except s of the primed particles of species $r_1, r_2, \ldots r_s$. Particle 1′ is the *same* particle as particle 1.

Now define δF_s by

$$\delta F_s^{r_1;r_2\cdots r_{s+1}}(X_1,t;X_2'\cdots X_{s+1}',t')$$
$$\equiv F_s^{r_1,r_2\cdots r_{s+1}}(X_1,t;X_2'\cdots X_{s+1},t') - f_1^{r_1}(X_1,t)\,f_s^{r_2\cdots r_{s+1}}(X_2'\cdots X_{s+1},t'). \tag{8}$$

Finally, we define our new hierarchy of correlation functions, Γ_s, by:

$$\Gamma_s^{r_0;r_1\cdots r_s}(X_0,t_0;X_1\cdots X_s,t)$$
$$\equiv \Big[\delta F_s^{r_0;r_1\cdots r_s}(X_0,t_0;X_1\cdots X_s,t) + \delta_{r_0r_1}\Omega_s^{r_0;r_1\cdots r_s}(X_0,t_0;X_1\cdots X_s,t)$$
$$+ \delta_{r_0r_2}\Omega_s^{r_0;r_2r_1r_3\cdots r_s}(X_0,t_0;X_2,X_1,X_3\cdots X_s,t) + \cdots$$
$$+ \delta_{r_0r_s}\Omega_s^{r_0;r_sr_2\cdots r_{s-1}r_1}(X_0,t_0;X_s,X_2,X_3\cdots X_{s+1}X_1,t)\Big]/f_1(X_0,t_0). \tag{9}$$

Ω_s and F_s are the test and field particle correlation functions introduced by Rostoker (1961). It will be shown that the combination Γ_s of the Rostoker functions Ω_s and F_s: (1) has a simple physical interpretation; (2) occurs naturally in the calculation of fluctuation spectra; and (3) satisfies surprisingly simple equations.

These distribution functions will first be interpreted as probabilities, and then the equations and initial conditions that they satisfy will be derived. These Rostoker functions may be interpreted in the following way. $f_s(X_1,\ldots X_s,t)$ is the probability density of finding particle 1 at X_1, 2 at $X_2,\ldots$ and s at X_s at time t. $\Omega_s(X_1,t;X_1',X_2',\ldots X_s',t')$ is the conditional probability that after finding particle 1 at X_1 at time t, the *same* particle is found at X_1' at time t' and, further, particles $2\cdots s$ are found at $X_2'\cdots X_s'$, respectively. $F_s(X_1,t;X_2',X_3',\ldots X_{s+1}',t')$ is the conditional probability that after finding particle 1 at X at time t the *different* particles $2\ldots s+1$ are found at $X_2,\ldots X_{s+1}'$, respectively, at time t'. $\delta F_s(X_1,t;X_2',\ldots X_{s+1}',t')$ is thus the change in the probability of finding particles $2 \cdots s+1$ at $X'_2 \cdots X'_{s+1}$ at t', given the information that particle 1 was at X_1 at time t. Ω_s is thus a test particle propagator, and δF_s is the response of the field particle propagator to the presence of the test particle. Finally, $\Gamma_s(X_0, t_0; X_1, X_2, \ldots X_s, t)$ may be interpreted as the propagator in which no account is taken as to whether the particles at $X_1, \ldots X_s$ are test particles (originating at X_0, t_0) or field particles. It includes both the test particle and the self-consistent response to the test particle.

The hierarchy of equations for f_s are obtained by integrating the Liouville equation (2) for $D_1(X,t)$ over all the X's except one, two, etc., successively, integrating by parts as necessary, and finally taking the thermodynamic limit $N_r, V \to \infty$, $N_r/V = n_r$, constant. The result of this procedure is the familiar Bogolubov–Born–Green–Kirkwood–Yvon (to be referred to as BBGKY) hierarchy (Yvon, 1935; Kirkwood, 1946, 1947; Born and Green, 1949; Green, 1952; Bogolubov, 1962).

$$\left(\frac{\partial}{\partial t} + \sum_{i=1}^{s} \boldsymbol{v}_i \cdot \frac{\partial}{\partial \boldsymbol{x}_i} + \sum_{i=1}^{s} \frac{e_{r_i}}{m_{r_i} c} \boldsymbol{v}_i \times \boldsymbol{B}_0 \cdot \frac{\partial}{\partial \boldsymbol{v}_i} - \sum_{\substack{i,j=1\\ i \neq j}}^{s} \frac{e_{r_i} e_{r_j}}{m_{r_i}} \frac{\partial}{\partial \boldsymbol{x}_i} \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} \cdot \frac{\partial}{\partial \boldsymbol{v}_i} \right) f_s^{r_1 \cdots r_s}(X_1 \cdots X_s, t)$$
$$= \sum_{i=1}^{s} \frac{e_{r_i}}{m_{r_i}} \sum_{r'} n_{r'} e_{r'} \int \mathrm{d}X' \frac{\partial}{\partial \boldsymbol{x}_i} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \cdot \frac{\partial}{\partial \boldsymbol{v}_i} f_{s+1}^{r_1 \cdots r_s, r'}(X_1, X_2 \cdots X_s, X', t). \tag{10}$$

The equations for the Rostoker functions Ω_s and F_s are obtained by integrating the Liouville equation for $D_2(Y_0, t_0; Y, t)$ over all the Y_0 variables except X_0, and over successively fewer of the Y variables. One obtains for the test particle correlations Ω_s:

$$\left[\frac{\partial}{\partial t} + \sum_{i=1}^{s} \left(\boldsymbol{v}_i \cdot \frac{\partial}{\partial \boldsymbol{x}_i} + \frac{e_{r_i}}{m_{r_i} c} \boldsymbol{v}_i \times \boldsymbol{B}_0 \cdot \frac{\partial}{\partial \boldsymbol{v}_i} \right) - \sum_{\substack{i,j=1\\ i \neq j}}^{s} \frac{e_{r_i} e_{r_j}}{m_{r_i}} \frac{\partial}{\partial \boldsymbol{x}_i} \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} \right] \cdot \frac{\partial}{\partial \boldsymbol{v}_i} \Omega_s^{r_0; r_1 \cdots r_s}(X_{10}, t_0; X_1 \cdots X_s, t)$$
$$= \sum_{i=1}^{s} \frac{e_{r_i}}{m_{r_i}} \sum_{r'} n_{r'} e_{r'} \int \mathrm{d}X' \frac{\partial}{\partial \boldsymbol{x}_i} \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}'|} \cdot \frac{\partial}{\partial \boldsymbol{v}_i} \Omega_{s+1}^{r_1; r_1 \cdots r_s, r'}(X_{10}, t_0; X_1 \cdots X_s, X', t), \tag{11}$$

and

$$\left[\frac{\partial}{\partial t} + \sum_{i=2}^{s+1} \left(\boldsymbol{v}_i \cdot \frac{\partial}{\partial \boldsymbol{x}_i} + \frac{e_{r_i}}{m_{r_i} c} \boldsymbol{v}_i \times \boldsymbol{B}_0 \cdot \frac{\partial}{\partial \boldsymbol{v}_i} \right) - \sum_{\substack{i,j=2\\ i \neq j}}^{s+1} \frac{e_{r_i} e_{r_j}}{m_{r_i}} \frac{\partial}{\partial \boldsymbol{x}_i} \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} \cdot \frac{\partial}{\partial \boldsymbol{v}_i} \right] \cdot F_s^{r_1; r_2 \cdots r_{s+1}}(X_{10}, t_0; X_2 \cdots X_{s+1}, t)$$
$$= n_{r_1} e_{r_1} \sum_{i=2}^{s+1} \frac{e_{r_i}}{m_{r_i}} \int \mathrm{d}X_i \frac{\partial}{\partial \boldsymbol{x}_i} \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}_1|} \cdot \frac{\partial}{\partial \boldsymbol{v}_i} \Omega_{s+1}^{r_1; r_1 \cdots r_{s+1}}(X_{10}, t_0; X_1, \ldots X_{s+1}, t)$$
$$+ \sum_{r'} n_{r'} e_{r'} \sum_{i=2}^{s+1} \frac{e_{r_i}}{m_{r_i}} \int \mathrm{d}X' \frac{\partial}{\partial \boldsymbol{x}_i} \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}'|} \cdot \frac{\partial}{\partial \boldsymbol{v}_i} F_{s+1}^{r_1; r_2 \cdots r_{s+1}, r'}(X_{10}, t_0; X_2 \cdots X_{s+1}, X', t), \tag{12}$$

for the field particle correlations, F_s. From (10) and (12) the equations for the field particle perturbations δF_s, which are the same as those for the F_s, are readily obtained:

$$\left[\frac{\partial}{\partial t} + \sum_{i=2}^{s+1} \left(\boldsymbol{v}_i \cdot \frac{\partial}{\partial \boldsymbol{x}_i} + \frac{e_{r_i}}{m_{r_i} c} \boldsymbol{v}_i \times \boldsymbol{B}_0 \cdot \frac{\partial}{\partial \boldsymbol{v}_i} \right) - \sum_{\substack{i,j=2\\ i \neq j}}^{s+1} \frac{e_{r_i} e_{r_j}}{m_{r_i}} \frac{\partial}{\partial \boldsymbol{x}_i} \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} \cdot \frac{\partial}{\partial \boldsymbol{v}_i} \right] \cdot \delta F_s^{r_1, r_2 \cdots r_{s+1}}(X_{10}, t_0, X_2 \cdots X_{s+1}, t)$$
$$= n_{r_1} e_{r_1} \sum_{i=2}^{s+1} \frac{e_{r_i}}{m_{r_i}} \int \mathrm{d}X_1 \frac{\partial}{\partial \boldsymbol{x}_i} \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}_1|} \cdot \frac{\partial}{\partial \boldsymbol{v}_i} \Omega_{s+1}^{r_1; r_1 \cdots r_{s+1}}(X_{10}, t_0; X_1 \cdots X_{s+1}, t)$$
$$+ \sum_{r'} n_{r'} e_{r'} \sum_{i=2}^{s+1} \frac{e_{r_i}}{m_{r_i}} \int \mathrm{d}X' \frac{\partial}{\partial \boldsymbol{x}_i} \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}'|} \cdot \frac{\partial}{\partial \boldsymbol{v}_i} \delta F_{s+1}^{r_1; r_2 \cdots r_{s+1}, r'}(X_{10}, t_0; X_2 \cdots X_{s+1}, X', t). \tag{13}$$

Combining the equations for Ω_s and δF_s (13), and using the definition (9) of Γ's enables the equations for the Γ's to be written as:

$$\left[\frac{\partial}{\partial t} + \sum_{i=1}^{s}\left(v_i\cdot\frac{\partial}{\partial x_i} + \frac{e_{r_i}}{m_{r_i}c}\, v_i \times B_0 \cdot \frac{\partial}{\partial v_i}\right) - \sum_{\substack{i,j=1\\ i\neq j}}^{s} \frac{e_{r_i}e_{r_j}}{m_{r_i}} \frac{\partial}{\partial x_i}\frac{1}{|x_i-x_j|}\cdot\frac{\partial}{\partial v_i}\right]$$

$$\cdot \Gamma_s^{r_0, r_1\cdots r_s}(X_0,t_0;X_1\cdots X_s,t)$$

$$= \sum_{r'} n_{r'}e_{r'}\sum_{i=1}^s \frac{e_{r_i}}{m_{r_i}}\int \mathrm{d}X' \frac{\partial}{\partial x_i}\frac{1}{|x_i-x'|}$$

$$\cdot\frac{\partial}{\partial v_i}\Gamma_{s+1}^{r_0;r_1\cdots r_s r'}(X_0,t_0;X_1\cdots X_s,X',t). \tag{14}$$

The Γ_s satisfy the BBGKY hierarchy! This is perhaps somewhat remarkable, as δF_s and Ω_s certainly do not. The reason lies in the fact that in our definition of Γ_s the information as to which specific particle was originally at the position X_0 at the time t_0 has been sacrificed. In a moment a definition will be given of Γ_s in terms of an expectation of a product of Klimontovich (1957) phase-space densities, where this result becomes transparent.

Having shown that our new hierarchy of functions satisfy the BBGKY hierarchy, their initial conditions, which follow from the initial conditions on D_2, (3), will next be determined. Using this and the definitions (7), (6), (8), and (9) of Ω_s, F_s, δF_s, and Γ_s respectively, the following are obtained:

$$\Omega_s^{r_1;r_1\cdots r_s}(X_{10},t_0;X_1\cdots X_s,t_0) = f_s^{r_1\cdots r_s}(X_1\cdots X_s,t_0)\,\delta(X_1-X_{10})/n_{r_1} \tag{15}$$

$$F_s^{r_1;r_2\cdots r_{s+1}}(X_{10},t_0;X_2\cdots X_{s+1},t_0) = f_{s+1}^{r_1\cdots r_{s+1}}(X_{10},X_2\cdots X_{s+1},t_0), \tag{16}$$

$$\delta F_s^{r_1;r_2\cdots r_{s+1}}(X_{10},t_0;X_2\cdots X_{s+1},t_0) = f_{s+1}^{r_1\cdots r_{s+1}}(X_{10},X_2\cdots X_{s+1},t_0) - f_1^{r_1}(X_{10},t_0)\,f_s^{r_2\cdots r_{s+1}}(X_2\cdots X_{s+1},t_0), \tag{17}$$

$$\Gamma_s^{r_0;r_1\cdots r_s}(X_0,t_0;X_1\cdots X_s,t_0) = \frac{f_{s+1}^{r_0\cdots r_s}(X_0\cdots X_s,t_0)}{f_1^{r_0}(X_0,t_0)} - f_s^{r_1\cdots r_s}(X_1\cdots X_s,t_0) + \sum_{i=1}^{s} \delta_{r_0 r_i}\,\delta(X_i-X_0)\,\frac{1}{n_{r_0}}\,\frac{f_s^{r_1\cdots r_s}(X_1,\ldots X_s,t_0)}{f_1(X_0,t_0)}. \tag{18}$$

It will now be shown that the correlation function of any "one-particle" [see (21)] macroscopic quantity such as density, velocity, or electric field, can be written in terms of Γ_1 and f_1 alone. It is not necessary to know δF_1 and Ω_1 separately.

Expectations

The *ensemble average* of any phase function $\psi[X;Y(t)]$, defined in six dimensional μ-space, is given by:

$$\langle\psi(X,t)\rangle \equiv \int \mathrm{d}Y D_1(Y,t)\,\psi[X;Y(t)] \tag{19}$$

$$= \int \mathrm{d}Y_0 D_1(Y_0,t_0)\,\psi\{X;Y[Y_0(t_0)]\}, \tag{20}$$

by Liouville's theorem. ψ is said to be a one-particle function if it can be written in the form

$$\psi[X;Y(t)] = \sum_r \psi_r(X,t)\sum_{i=1}^{N_r}\delta[X-X_i(t)]. \tag{21}$$

If a one-particle function $\psi[X;Y(t)]$ is defined by

$$\psi[X;Y(t)] \equiv \sum_r\sum_i \phi_r[X;X_i(t)],$$

then the operator $\psi_r(x,t)$ is given by

$$\psi_r(X,t) = \int \mathrm{d}X'\phi_r(X;X'),$$

where $X_i^r(t)$ is the exact trajectory of the ith particle of species r. For example, when $\psi_r = \delta_{rr'}$, ψ is the phase-space microdensity of the r' species, then

$$\langle\psi(X,t)\rangle = n_{r'}f_1^{r'}(X,t), \tag{22}$$

as may be readily verified by substituting (21) into (19) and integrating.

Similarly, any two-time phase function $\psi_2[X_0,Y_0(t);X,Y(t)]$ has its ensemble average defined by:

$$\langle\psi_2(X_0,t_0;X,t)\rangle = \int \mathrm{d}Y_0\mathrm{d}Y D_2(Y_0,t_0;Y,t)\,\psi_2[X_0,Y_0(t_0);X,Y(t)]. \tag{23}$$

The correlation function of fluctuations of a one particle operator ψ is thus given by

$$\langle\{\psi[X_0;Y(t_0)]-\langle\psi[X_0;Y(t_0)]\rangle\}\{\psi[X,Y(t)]-\langle\psi[X;Y(t)]\rangle\}\rangle$$

$$= \langle\psi[X_0;Y(t_0)]\psi[X;Y(t)]\rangle - \langle\psi[X_0;Y(t_0)]\rangle\langle\psi[X,Y(t)]\rangle$$

$$= \sum_{r,r_0} n_r r_0\psi_{r_0}(X_0,t_0)\psi_r(X,t)\left[\delta_{rr_0}\Omega_1^{r_0;r}(X_0,t_0;X,t) + F_1^{r_0;r}(X_0,t_0;X,t)\right]$$

$$-\left(\sum_{r_0} n_{r_0}\psi_{r_0}(X_0,t_0)f_1^{r_0}(X_0,t_0)\right)\left(\sum_r n_r\psi_r(X,t)f_1^r(X,t)\right)$$

$$= \sum_{r,r_0} n_{r_0}n_r\psi_{r_0}(X_0,t_0)\psi_r(X,t)f_1^{r_0}(X_0,t_0)\Gamma_1^{r_0;r}(X_0,t_0;X,t). \tag{24}$$

To give an example of the application of (24), typical macroscopic quantities are the ensemble average number density, momentum density, and kinetic energy density of the rth species, when

$$\psi_{r'}(X,t)=\delta_{rr'}\int \mathrm{d}^3v\, s_i(\boldsymbol{v}), \tag{25}$$

where

$$s_i(\boldsymbol{v})=1, m_r\boldsymbol{v}, \tfrac{1}{2}m_r v^2, \tag{26}$$

respectively. The time correlation of the fluctuations of these quantities are thus given by

$$n_{r_0}n_r\int \mathrm{d}^3v_0\, s_i(\boldsymbol{v}_0) f_1^{r_0}(X_0,t_0)\int \mathrm{d}^3v\, s_j(\boldsymbol{v})\Gamma_1^{r_0;r}(X_0,t_0;X,t). \tag{27}$$

To give one further useful example, the electric field time autocorrelation is given by

$$\langle \boldsymbol{E}(\boldsymbol{x}_0',t_0)\boldsymbol{E}(\boldsymbol{x}',t)\rangle=\sum_{r_0,r}\int \mathrm{d}X_0\,\mathrm{d}X f_1^{r_0}(X_0,t_0)\Gamma_1^{r_0,r}(X_0,t_0,X,t) \times\frac{e_{r_0}(\boldsymbol{x}_0'-\boldsymbol{x}_0)}{|\boldsymbol{x}_0'-\boldsymbol{x}_0|^3}\,\frac{e_r(\boldsymbol{x}'-\boldsymbol{x})}{|\boldsymbol{x}'-\boldsymbol{x}|^3}. \tag{28}$$

To reiterate, all one needs to compute the time correlations of any local (one-particle) macroscopic quantity is f_1 and Γ_1, the lowest members of the f and Γ hierarchies, respectively. In order to solve for f_1 or Γ_1 one must truncate the hierarchies by some suitable approximate scheme.

Much work has been done using the f hierarchy to develop a closed kinetic equation for f. To do this in a systematic fashion, one must estimate the various terms in the hierarchy equation (10), and identify some suitable small parameter in which to attempt an asymptotic solution. It is well known that there are three regimes in which such a parameter is available—the Boltzmann, the weak-coupling, and the plasma regimes—in which the small parameters are, respectively, the number of particles within the range of one-particle interaction potential, the interaction potential divided by the thermal energy, and the plasma parameter [defined to be the reciprocal of the number of particles within a sphere of radius equal to the Debye length $(n\lambda_{\mathrm{D}}^3)$]. We are concerned with the plasma regime, where the ordering of the terms in (14) is

$$1:1:\Omega_{\mathrm{e}}/\omega_{\mathrm{p}}:\varepsilon:1, \tag{29}$$

where $\Omega_{\mathrm{e}}\equiv eB/mc$ is the cyclotron frequency, $\omega_{\mathrm{p}}^2\equiv 4\pi ne^2/m$ is the plasma frequency squared, when typical lengths and times have been scaled to the Debye length and plasma period, respectively. The requirements for the plasma ordering are actually somewhat weaker than implied here. All that is needed is that the internal electrostatic field energy density be small compared with the particle energy, that is: $E^2/8\pi nT\ll 1$. To retain maximum generality, suppose that $\Omega_{\mathrm{e}}/\omega_{\mathrm{p}}$ is O(1): then strong or weak magnetic field limits may be recovered in a subsidiary expansion if desired. Note, however, that one may not obtain the results of guiding center theory by taking the limit of arbitrarily large magnetic field, as this involves an interchange of the order of the large magnetic field and plasma limits and this may be nonuniform.

Cluster expansions

First, note that the f_s hierarchy is satisfied identically in lowest (zeroth) order in ε by

$$f_s^{r_1\cdots r_s}(X_1\cdots X_s,t)=f_1^{r_1}(X_1,t)f_1^{r_1}(X_2,t)\cdots f_1^{r_s}(X_s,t), \tag{30}$$

provided that f satisfies the Vlasov equation:

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}+\frac{e_r}{m_rc}\boldsymbol{v}\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial \boldsymbol{v}}\right)f_1^r(X,t) =\frac{e_r}{m_r}\sum_{r'}n_{r'}e_{r'}\int \mathrm{d}X' f_1(X',t)\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\cdot\frac{\partial}{\partial \boldsymbol{v}}f_1(X,t). \tag{31}$$

Note also that the Γ_s hierarchy is identically satisfied in zeroth order by

$$\Gamma^{r_0;r_1\cdots r_s}(X_0,t_0;X_1\cdots X_3,t)=\Gamma_1^{r_0;r_1}(X_0,t_0;X_1,t)f_1^{r_2}(X_1,t)\cdots f_1^{r_s}(X_s,t) +f_1^{r_1}(X_1,t)\Gamma_1^{r_0;r_2}(X_0,t_0;X_2,t)f_1^{r_3}\cdots f_1^{r_s} +\cdots+f_1^{r_1}(X_1,t)\cdots f_1^{r_{s-1}}(X_{s-1},t)\Gamma_1^{r_0;r_s}(X_0,t_0;X_s,t), \tag{32}$$

provided that f satisfies the Vlasov equation (31), and Γ_1 satisfies the linearized Vlasov equation:

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}+\frac{e_r}{m_rc}\boldsymbol{v}\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial \boldsymbol{v}}\right)\Gamma_1(X_0,t_0;X,t) -\frac{e_r}{m_r}\sum_{r'}n_{r'}e_{r'}\int \mathrm{d}X\,\Gamma_1^{r_0;r'}(X_0,t_0;X',t)\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\cdot\frac{\partial}{\partial \boldsymbol{v}}f_1^r(X,t) -\frac{e_r}{m_r}\sum_{r'}n_{r'}e_{r'}\int \mathrm{d}X' f_1^r(X',t)\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\cdot\frac{\partial}{\partial \boldsymbol{v}}\Gamma_1^{r_0;r}(X_0,t_0;X,t)=0, \tag{33}$$

with singular initial condition from (18). This, together with the method's success in equilibrium statistical mechanics motivates the introduction of the Mayer and Mayer (1940) cluster expansion for f and a (linearized!) modification of it for Γ_s. That is,

$$\begin{aligned}
\bar{f}_1(1)&=f_1(1),\\
\bar{f}_2(1,2)&=f_2(1,2)-f_1(1)f_1(2),\\
\bar{f}_3(1,2,3)&=f_3(1,2,3)-f_1(1)f_2(2,3)-f_1(2)f_2(3,1)-f_1(3)f_2(1,2)\\
&\quad+2f_1(1)f_1(2)f_1(3),\\
\bar{f}_s(1,2,\ldots s)&=f_s(1,2,\ldots s)-\sum_{n,p}(-1)^n(n-1)!f_{p_1}f_{p_2}\cdots f_{p_n}.
\end{aligned} \tag{34}$$

The sum is over all partitions $p=(p_1,\dots p_n)$ of the set $1,2,\dots s$ into n subsets $p_1,\dots p_n$. $\bar f_2(1,2)$ is shorthand for $\bar f_2^{r_1 r_2}(X_1, X_2, t)$ etc. The sum $\sum p_n = s$.

Using the property that if the particles 1 through s are divided into groups $p_1,\dots p_n$, (this is equivalent to the apparently weaker assumption of reducibility on $p_1,\dots p_n$, the division into only two groups) and that the groups are widely separated in space, the distribution function f_s factorizes into the form $f_{p_1} f_{p_2}\cdots f_{p_n}$, one may show that the function $\bar f_s$ defined above (34) is irreducible, in the sense that it vanishes on any such partition and separation of the particles.

Solving (34) back for the distribution functions f_s in terms of their irreducible parts gives:

$$\begin{aligned}
f_1(1) &= \bar f_1(1),\\
f_2(1,2) &= \bar f_1(1)\bar f_1(2)+\bar f_2(1,2),\\
f_2(1,2,3) &= \bar f_1(1)\bar f_1(2)\bar f(3)+\bar f_1(1)\bar f_2(2,3)+\bar f_1(2)\bar f_2(3,1)\\
&\quad+\bar f_1(3)\bar f_2(1,2)+\bar f_3(1,2,3),\\
f_s(1,\dots,s) &= \sum_{n,p} \bar f_{p_1}\cdots\bar f_{p_n}.
\end{aligned}\tag{35}$$

Using the factorization property that

$$\Gamma_s(0;1,2,\dots s)\to\Gamma_{p_1}f_{p_2}f_{p_3},\dots f_{p_n}+f_{p_1}\Gamma_{p_2}f_{p_3},\dots f_{p_n}+\cdots+f_{p_1}f_{p_2},\dots f_{p_{n-1}}\Gamma_{p_n},\tag{36}$$

if the particles $1,\dots s$ are separated into n groups $p_1,\dots p_n$, one can show that the following defines the irreducible parts $\bar\Gamma_s$ of Γ_s [the initial arguments (X_0, t) will be omitted in (37) and (38) for compactness]:

$$\begin{aligned}
\bar\Gamma_1(1) &= \Gamma_1(1),\\
\bar\Gamma_2(1,2) &= \Gamma_2(1,2)-\Gamma_1(1)f_1(2)-\Gamma_1(2)f_1(1),\\
\bar\Gamma_3(1,2,3) &= \Gamma_3(1,2,3)-\Gamma_1(1)f_2(2,3)-\Gamma_1(2)f_2(3,1)-\Gamma_1(3)f_2(1,2)\\
&\quad-f_1(1)\Gamma_2(2,3)-f_1(2)\Gamma_2(3,1)-f_1(3)\Gamma_2(1,2)\\
&\quad+2\Gamma_1(1)f_1(2)f_1(3)+2f_1(1)\Gamma_1(2)f_1(3)+2f_1(1)f_1(2)\Gamma_1(3),\\
\bar\Gamma_s(1,2,\dots s) &= \Gamma_s(1,2,\dots)-\sum_{n,p}(-1)^n(n-1)!\big(\Gamma_{p_1}f_{p_2}\cdots f_{p_n}\\
&\quad+f_{p_1}\Gamma_{p_2}f_{p_3}\cdots f_{p_m}+\cdots+f_{p_1}\cdots f_{p_{n-1}}\Gamma_{p_n}\big).
\end{aligned}\tag{37}$$

Solving back for Γ_s gives:

$$\begin{aligned}
\Gamma_1(1) &= \bar\Gamma_1(1),\\
\Gamma_2(1,2) &= \bar\Gamma_1(1)\bar f_1(2)+\bar\Gamma_1(2)\bar f_1(1)+\bar\Gamma_2(1,2),\\
\Gamma_3(1,2,3) &= \bar\Gamma_1(1)\bar f_1(2)\bar f_1(3)+\bar f_1(1)\bar\Gamma_2(2,3)+\bar\Gamma_1(1)\bar f_2(2,3)\\
&\quad+(1\to2,2\to3,3\to1)+(1\to3,3\to2,2\to1),\\
\Gamma_s(1,2,\dots s) &= \sum_{n,p}\big(\bar\Gamma_{p_1}\bar f_{p_2}\cdots\bar f_{p_n}+\bar f_{p_1}\bar\Gamma_{p_2}\bar f_{p_3}\cdots\bar f_{p_n}+\cdots+\bar f_{p_1}\cdots\bar f_{p_{n-1}}\bar\Gamma_{p_n}\big).
\end{aligned}\tag{38}$$

The equations for $\bar f_s$ and $\bar\Gamma_s$ may now be written using the cluster expansions (35) and (38), and the BBGKY hierarchy (10) and (14). Only the equations up to order ε^2 will be needed, that is, for $\bar f_1$, $\bar f_2$, $\bar f_3$, $\bar\Gamma_1$, and $\bar\Gamma_2$. The low-order irreducible distribution functions will be renamed from here on.

$$\begin{aligned}
&\bar f_1\to f, &&\bar f_2\to g, &&\bar f_3\to h,\\
&\bar\Gamma_1\to\Gamma, &&\bar\Gamma_2\to\Delta, &&\bar\Gamma_3\to\varepsilon.
\end{aligned}\tag{39}$$

The cluster functions f, g, h, Γ, and Δ then satisfy the following equations:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}+\frac{e_r}{m_r c}\boldsymbol{v}\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial\boldsymbol{v}}\right)f^r(X,t)\\
&\quad-\frac{e_r}{m_r}\frac{\partial f^r}{\partial\boldsymbol{v}}(X,t)\cdot\sum_{r'}n_{r'}e_{r'}\int \mathrm{d}X' f^{r'}(X',t)\frac{\partial}{\partial\boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\\
&\quad=\frac{e_r}{m_r}\sum_{r'}n_{r'}e_{r'}\int\mathrm{d}X'\frac{\partial}{\partial\boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\cdot\frac{\partial}{\partial\boldsymbol{v}}g^{rr'}(X,X',t),
\end{aligned}\tag{40}$$

$$\begin{aligned}
&\Bigg[\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial\boldsymbol{x}}+\boldsymbol{v}'\cdot\frac{\partial}{\partial\boldsymbol{x}}+\frac{e_r}{m_r c}\boldsymbol{v}\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial\boldsymbol{v}}+\frac{e_{r'}}{m_{r'}c}\boldsymbol{v}'\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial\boldsymbol{v}'}\\
&\quad-e_re_{r'}\frac{\partial}{\partial\boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\cdot\left(\frac{1}{m_r}\frac{\partial}{\partial\boldsymbol{v}}-\frac{1}{m_{r'}}\frac{\partial}{\partial\boldsymbol{v}'}\right)\Bigg]g(X,X',t)\\
&\quad=\sum_{r''}n_{r''}e_{r''}\int\mathrm{d}X''\Bigg[\bigg(\frac{e_r}{m_r}\frac{\partial}{\partial\boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}''|}\cdot\frac{\partial}{\partial\boldsymbol{v}}[f(X,t)g(X',X'',t)\\
&\quad+f(X'',t)g(X,X',t)+h(X,X',X'',t)]\bigg)+\{X\leftrightarrow X'\}\Bigg]\\
&\quad+e_re_{r'}\frac{\partial}{\partial\boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\cdot\left(\frac{1}{m_r}\frac{\partial}{\partial\boldsymbol{v}}-\frac{1}{m_{r'}}\frac{\partial}{\partial\boldsymbol{v}'}\right)f(X,t)f(X',t),
\end{aligned}\tag{41}$$

$$\begin{aligned}
&\Bigg[\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial\boldsymbol{x}}+\boldsymbol{v}'\cdot\frac{\partial}{\partial\boldsymbol{x}'}+\boldsymbol{v}''\cdot\frac{\partial}{\partial\boldsymbol{x}''}+\frac{e_r}{m_r c}\boldsymbol{v}\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial\boldsymbol{v}}+\frac{e_{r'}}{m_{r'}c}\boldsymbol{v}'\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial\boldsymbol{v}'}\\
&\quad+\frac{e_{r''}}{m_{r''}c}\boldsymbol{v}''\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial\boldsymbol{v}''}-e_re_{r'}\frac{\partial}{\partial\boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\cdot\left(\frac{1}{m_r}\frac{\partial}{\partial\boldsymbol{v}}-\frac{1}{m_{r'}}\frac{\partial}{\partial\boldsymbol{v}'}\right)\\
&\quad-e_{r'}e_{r''}\frac{\partial}{\partial\boldsymbol{x}'}\frac{1}{|\boldsymbol{x}'-\boldsymbol{x}''|}\left(\frac{1}{m_{r'}}\frac{\partial}{\partial\boldsymbol{v}'}-\frac{1}{m_{r''}}\frac{\partial}{\partial\boldsymbol{v}''}\right)\\
&\quad-e_{r''}e_r\frac{\partial}{\partial\boldsymbol{x}''}\frac{1}{|\boldsymbol{x}''-\boldsymbol{x}|}\cdot\left(\frac{1}{m_{r''}}\frac{\partial}{\partial\boldsymbol{v}''}-\frac{1}{m_r}\frac{\partial}{\partial\boldsymbol{v}}\right)\Bigg]h(X,X',X'',t)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{r''} n_{r'''} e_{r'''} \int \mathrm{d}X''' \bigg[\bigg(\frac{e_r}{m_r}\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'''|}\cdot\frac{\partial}{\partial \boldsymbol{v}}[f(X,t)h(X',X'',X''',t)\\
&\quad + g(X,X',t)g(X'',X''',t)+g(X,X'',t)g(X',X''',t)\\
&\quad + f(X''',t)h(X,X',X'',t)+k(X,X',X'',X''',t)]\bigg)+\{X'\leftrightarrow X\}+\{X''\leftrightarrow X\}\bigg]\\
&\quad + e_r e_{r'} \frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\cdot\left(\frac{1}{m_r}\frac{\partial}{\partial \boldsymbol{v}}-\frac{1}{m_{r'}}\frac{\partial}{\partial \boldsymbol{v}'}\right)\\
&\quad \cdot[f(X,t)g(X',X'',t)+f(X',t)g(X'',X,t)]\\
&\quad + \text{cyclic perm}\,(X'\to X\to X''\to X')+\text{cyclic perm}\,(X'\to X''\to X\to X'), \qquad (42)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}+\frac{e_r}{m_r c}\boldsymbol{v}\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial \boldsymbol{v}}\right)\Gamma(X_0,t_0;X,t)\\
&\quad -\frac{e_r}{m_r}\frac{\partial f(X,t)}{\partial \boldsymbol{v}}\cdot\sum_{r'} n_{r'}e_{r'}\int \mathrm{d}X'\,\Gamma(X_0,t_0;X',t)\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\\
&\quad -\frac{e_r}{m_r}\frac{\partial}{\partial \boldsymbol{v}}\Gamma(X_0,t_0;X,t)\sum_{r'} n_{r'}e_{r'}\int \mathrm{d}X'\,f(X',t)\cdot\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\\
&= \frac{e_r}{m_r}\sum_{r'} n_{r'}e_{r'}\int \mathrm{d}X'\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\cdot\frac{\partial}{\partial \boldsymbol{v}}\Delta(X_0,t_0;X,X',t), \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
&\bigg[\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}+\boldsymbol{v}'\cdot\frac{\partial}{\partial \boldsymbol{x}'}+\frac{e_r}{m_r c}\boldsymbol{v}\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial \boldsymbol{v}}+\frac{e_{r'}}{m_{r'} c}\boldsymbol{v}'\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial \boldsymbol{v}'}\\
&\quad - e_r e_{r'}\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\cdot\left(\frac{1}{m_r}\frac{\partial}{\partial \boldsymbol{v}}-\frac{1}{m_{r'}}\frac{\partial}{\partial \boldsymbol{v}'}\right)\bigg]\Delta(X_0,t_0;X,X',t)\\
&= e_r e_{r'}\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\cdot\left(\frac{1}{m_r}\frac{\partial}{\partial \boldsymbol{v}}-\frac{1}{m_{r'}}\frac{\partial}{\partial \boldsymbol{v}'}\right)\\
&\quad \cdot[\Gamma(X_0,t_0,X,t)f(X',t)+\Gamma(X_0,t_0;X',t)f(X,t)]\\
&\quad + \sum_{r''} n_{r''}e_{r''}\int \mathrm{d}X''\bigg[\bigg(\frac{e_r}{m_r}\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}''|}\frac{\partial}{\partial \boldsymbol{v}}[f(X,t)\Delta(X_0,t_0;X',X'',t)\\
&\quad + f(X'',t)\Delta(X_0,t_0;X,X',t)+\Gamma(X_0,t_0;X'',t)g(X,X',t)\\
&\quad + \Gamma(X_0,t_0;X,t)g(X',X'',t)+\varepsilon(X_0,t_0;X,X',X'',t)]\bigg)+\{X\leftrightarrow X'\}\bigg]. \qquad (44)
\end{aligned}
$$

The species labels on the distribution functions have been omitted. Here, and in the future, they may be implied from the context in the following manner. Any function of the form $\chi(X_1\cdots X_s,t)$ is understood to be $\chi^{r_1\cdots r_s}(X_1\cdots X_s,t)$, and any function of the form $\chi(X_0,t_0:X_1\cdots X_s,t)$ is understood to be $\chi^{r_0;r_1\cdots r_s}(X_0,t_0:X_1\cdots X_s,t)$, where the species labels always correspond to the phase space arguments of the functions.

From the initial conditions on Γ_s, (18), one may readily show that $\bar{\Gamma}_s$ has the following initial condition

$$
\begin{aligned}
\bar{\Gamma}_s(X_0,t_0;\Psi_1,X_2\cdots X_s,t_0) &= \frac{\bar{f}_{s+1}(X_0,X_1\cdots X_s,t_0)}{\bar{f}_1(X_0,t_0)}\\
&\quad + \frac{\bar{f}_s(X_1\cdots X_s,t_0)}{\bar{f}_1(X_0,t_0)}\frac{1}{n_{r_0}}\sum_{n=1}^{s}\delta_{r_0 r_i}\delta(X_i-X_0). \qquad (45)
\end{aligned}
$$

The simplicity of this result derives from the initial conditions on $\bar{\Gamma}_s$ having to be irreducible. The cases of interest to use are the following:

$$
\Gamma_1(X_0,t_0;X_1,t_0) = \frac{g(X_0,X_1,t_0)}{f(X_0,t_0)}+\delta_{rr_0}\frac{1}{n_{r_0}}\delta(X_1-X_0), \qquad (46)
$$

$$
\begin{aligned}
\Delta(X_0,t_0;X_1,X_2,t_0) &= \frac{h(X_0,X_1,X_2,t_0)}{f(X_0,t_0)}\\
&\quad + \frac{g(X_1,X_2,t_0)}{f(X_0,t_0)}\frac{1}{n_{r_0}}\left[\delta_{r_0 r_1}\delta(X_1-X_0)+\delta_{r_0 r_2}\delta(X_2-X_0)\right], \qquad (47)
\end{aligned}
$$

Equation (45) may be interpreted physically as saying that at time t_0 the particle at X_1 was a test particle, giving rise to the delta-function term, or was a member of its shield cloud, giving rise to the g term. A similar interpretation of (46) can be made. It is clear from this interpretation how Γ_1 can be regarded as the propagator of a dressed test particle. The initially singular part describes the bare particle, and the initially smooth part describes the dressing.

The relative ordering of the terms in the f, g, and h equations are as follows:

$$
\begin{aligned}
&1:1:\Omega_e/\omega_p:0:\varepsilon,\\
&1:1:1:\Omega_e/\omega_p:\Omega_e/\omega_p:\varepsilon:[1:0:\varepsilon]:[1:0:\varepsilon]:1,\\
&1:1:1:1:\Omega_e/\omega_p:\Omega_e/\omega_p:\Omega_e/\omega_p:\varepsilon:\varepsilon:\varepsilon:([1:1:1:\varepsilon]:[1:1:1:\varepsilon]:[1:1:1:\varepsilon]:1:1:), \qquad (48)
\end{aligned}
$$

respectively. The usual estimate has been used that $\bar{f}_s = \mathrm{O}(\varepsilon^{s-1})$. The relative ordering of the terms in the Γ and Δ equations are

$$
\begin{aligned}
&1:1:\Omega_e/\omega_p:1:0:\varepsilon,\\
&1:1:1:\Omega_e/\omega_p:\Omega_e/\omega_p:\varepsilon:1:1:[1:0:1:1:\varepsilon]:[1:0:1:1:\varepsilon]. \qquad (49)
\end{aligned}
$$

Terms ordered as 0 vanish in the homogeneous limit, and are otherwise small in the Debye length over the inhomogeneity scale length. The appropriate ordering of the $\bar{\Gamma}_s$ consistent with (49) and the initial conditions (45) is $\bar{\Gamma}_s = O(\varepsilon^s)$. Note that the order of $\bar{\Gamma}_s$ is one higher than the corresponding $\bar{f}_s$, because knowledge of the initial condition on one particle makes an $O(\varepsilon)$ perturbation to the background. Using this ordering in the plasma parameter ε, we can truncate the hierarchy of equations for the $\bar{f}_s$ and $\bar{\Gamma}_s$ by working to any desired (fixed) order in ε. Rostoker worked to order ε. After rederiving Rostoker's lowest-order results, we will work to order ε^2, where the basic equations of the theory are (41) and (43) and the following:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}} + \boldsymbol{v}'\cdot\frac{\partial}{\partial \boldsymbol{x}'} + \frac{e_r}{m_r c}\boldsymbol{v}\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial \boldsymbol{v}} + \frac{e_{r'}}{m_{r'} c}\boldsymbol{v}'\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial \boldsymbol{v}'}\right) g(X, X', t) \\ = e_r e_{r'}\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\left(\frac{1}{m_r}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_{r'}}\frac{\partial}{\partial \boldsymbol{v}'}\right) f_1(X,t) f_1(X',t) \\ + \sum_{r''} n_{r''} e_{r''}\int \mathrm{d}X''\bigg(\frac{e_r}{m_r}\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}''|}\cdot\frac{\partial}{\partial \boldsymbol{v}}[f(X,t)g(X',X'',t) \\ + f(X'',t)g(X,X',t)] + (X' \leftrightarrow X)\bigg),\end{aligned} \tag{50}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}} + \boldsymbol{v}'\cdot\frac{\partial}{\partial \boldsymbol{x}'} + \frac{e_r}{m_r c}\boldsymbol{v}\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial \boldsymbol{v}} + \frac{e_{r'}}{m_{r'} c}\boldsymbol{v}'\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial \boldsymbol{v}'}\right) \Delta(X_0, t_0; X, X', t) \\ = e_r e_{r'}\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\cdot\left(\frac{1}{m_r}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_{r'}}\frac{\partial}{\partial \boldsymbol{v}'}\right) \\ \cdot[\Gamma_1(X_0,t_0;X,t) f(X',t) + \Gamma_1(X_0,t_0;X',t) f(X,t)] \\ + \sum_{r''} m_{r''} e_{r''}\int \mathrm{d}X''\bigg(\frac{e_r}{m_r}\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}''|}\frac{\partial}{\partial \boldsymbol{v}}[f_1(X,t)\Delta(X_0,t_0;X',X'',t) \\ + f(X'',t)\Delta(X_0,t_0;X,X',t) + \Gamma(X_0,t_0;X',t)g(X,X',t) \\ + \Gamma(X_0,t_0;X,t)g(X',X'',t)] + (X \leftrightarrow X')\bigg),\end{aligned} \tag{51}$$

together with the initial conditions (47).

It is useful to note as a formal device, that to obtain the equations for Γ, Δ, and ε, etc. from the more familiar equations for f, g, and h etc., one may formally replace f by $f+\Gamma$, g by $g+\Delta$, h by $h+\varepsilon$ in the equations for f, g, and h, regard Γ, Δ, and ε as small perturbations, and collect the terms that are first-order in these small quantities. One then obtains precisely the equations for Γ, Δ, and ε.

It is worth pointing out that our approach to the theory of fluctuations through the two-time hierarchy, as opposed to a phenomenological or a thermodynamic approach, at least starts with exact equations, and even though approximations are necessary to solve them, error estimates are at least in principle obtainable.

The Γ hierarchy in the Klimontovich formalism

The *regular hierarchy functions* may be defined as expectation values of products of the Klimontovich phase space density function $N_r(X,t)$, defined by

$$N_r(X,t) = \frac{1}{n_r}\sum_{i=1}^{N_r}\delta[X - X_i^r(t)], \tag{52}$$

where $X_i^r(t)$ is the exact trajectory of the ith particle of the rth species. $f_i^r(X,t)$ is just the expectation value of $N_r(X,t)$, that is

$$\langle N_r(X,t)\rangle = \frac{1}{n_r}\int \mathrm{d}Y\, D_1(Y,t) N_r = f_1^r(X,t) \tag{53}$$

If a subtraction operator S is defined, which, when acting on any product of N_r's removes all terms containing products of delta functions of the same particle at different points, for example

$$\begin{aligned} SN_r(X,t)N_{r'}(X',t) &= \frac{1}{n_r^2}\sum_{\substack{i,j=1\\ i\neq j}}^{N_r}\delta[X - X_i^r(t)]\delta[X' - X_j^r(t)], & r = r' \\ &= \frac{1}{n_r n_{r'}}\sum_{i=1}^{N_r}\sum_{j=1}^{N_{r'}}\delta[X - X_i^r(t)]\delta[X' - X_j^{r'}(t)], & r \neq r' \end{aligned} \tag{54}$$

then it may readily be shown that

$$\langle SN_{r_1}(X_1,t)N_{r_2}(X_2,t)\cdots N_{r_s}(X_s,t)\rangle = f_1^{r_1\cdots r_s}(X_1, X_2\cdots X_s,t). \tag{55}$$

From the equation of motion for $N_r(X,t)$

$$\begin{aligned} 0 = \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}} + \frac{e_r}{m_r c}\boldsymbol{v}\times\boldsymbol{B}_0\cdot\frac{\partial}{\partial \boldsymbol{v}}\right) N_r(X,t) \\ - \frac{e_r}{m_r}\sum_{r'} n_{r'} e_{r'}\int \mathrm{d}X'\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\cdot\frac{\partial}{\partial \boldsymbol{v}} SN_r(X,t)N_{r'}(X',t) \end{aligned} \tag{56}$$

proved by noting that

$$\frac{\partial}{\partial t}\delta[X - X_i^r(t)] = -\frac{\partial X_i^r(t)}{\partial t}\frac{\partial}{\partial X}\delta[X - X_i^r(t)] \tag{57}$$

and that the subtraction operator arises from the lack of electrostatic self forces, it may be proved that $SN_{r_1}(X_1,t)N_{r_2}(X_2,t)\cdots N_{r_s}(X_s,t)$ satisfies the BBGKY hierarchy. On taking expectations with $D_1(Y,t)$ this provides an alternative derivation of the hierarchy for f_s. It is clear that $N_{r_0}(X_0,t_0)SN_{r_1}(X_1,t)N_{r_2}(X_2,t)\cdots N_{r_s}(X_s,t)$ must also satisfy the BBGKY hierarchy [in the variables (X,t)], as $N_{r_0}(X_0,t_0)$ commutes with all the differential and integral operators. The same must be true of its expectation taken with $D_2(Y_0, t_0, Y, t)$. Finally, the quantity

$$\begin{aligned}\langle N_{r_0}(X_0,t_0) S N_{r_1}(X_1,t) N_{r_2}(X_2,t) \cdots N_{r_s}(X_s,t)\rangle / \langle N_{r_0}(X_0,t_0)\rangle \\ -\langle S N_{r_1}(X_1,t) N_{r_2}(X_2,t) \cdots N_{r_s}(X_s,t)\rangle \end{aligned} \tag{58}$$

must also satisfy the same hierarchy, but a straightforward calculation shows this quantity to be none other than

$$\Gamma_s^{r_0; r_1 \cdots r_s}(X_0, t_0; X_1 \cdots X_s, t) \tag{59}$$

An alternative approach for deriving the formalism of Section 2.3.2 would have been to have *defined* Γ_s by the expression (58), on the grounds that it transparently satisfies the BBGKY hierarchy, but this was rejected in favor of the development in terms of the more familiar Rostoker functions which shows their probabilistic interpretation.

2.3.3. Fluctuations in uniform plasma

Introduction

In this section the spectrum of fluctuations of a uniform plasma will be found in various approximations, where frequencies are of the order of the plasma frequency $\omega \gtrsim \omega_p$, or, more properly, well above collision frequencies. Wavelengths will be considered where the collisional damping is dominant over the Landau damping, as well as those where Landau damping dominates. The case of low-frequency and hydrodynamic fluctuations will be treated in Sections 2.3.4 and 2.3.5.

The electron charge density fluctuation spectrum is important experimentally, as it determines the incoherent scatter of an electromagnetic wave from a plasma. The incoherent differential scattering cross section through wavevector $\boldsymbol{k}$ with frequency ω is proportional to the spectral intensity of electron charge density fluctuations of wavevector $(\boldsymbol{K}-\boldsymbol{k})$ and frequency $(\Omega-\omega)$ where $\boldsymbol{K}$ and Ω are the wavevector and frequency of the incident wave. The explanation of the ionospheric incoherent radar backscatter experiments of Bowles (1958) was a major success of plasma kinetic theory, and at the time it was one of the few meeting points of theory and experiment.

Lowest-order theory of fluctuations of a stable uniform plasma

The results of this section have previously been obtained by Rostoker (1961). Its purpose is to establish the connection between the formalisms. First conventions are established for Fourier and Laplace transforms, and the spectral intensity of a fluctuating quantity is defined.

For any function of position $f(\boldsymbol{x})$, its *Fourier transform* $f(\boldsymbol{k})$ is defined by:

$$f(\boldsymbol{k}) = \int \mathrm{d}^3x\, f(\boldsymbol{x}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \tag{60}$$

with inversion:

$$f(\boldsymbol{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{61}$$

For any function of time $f(t)$ its one-sided Fourier (Laplace) transform is defined by:

$$f(\omega) = \int_0^\infty f(t) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t \tag{62}$$

with inversion:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}\omega \tag{63}$$

with the contour running above all the singularities of $f(\omega)$ in the complex ω plane.

Supposing $y(t)$ is a stationary random process, its autocorrelation is defined by:

$$C(t) = \langle y(t_0) y(t_0+\tau)\rangle. \tag{64}$$

The average is an ensemble average. One of the theses of equilibrium statistical mechanics is the *ergodic hypothesis*, which would equate this to the time average over t_0. Its spectral intensity is defined by:

$$S(\omega) = \int_{-\infty}^{\infty} C(\tau) \mathrm{e}^{\mathrm{i}\omega\tau} \mathrm{d}\tau \tag{65}$$

so that

$$C(\tau) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} S(\omega) \mathrm{e}^{-\mathrm{i}\omega\tau}. \tag{66}$$

Since Laplace transforms will be involved, $S(\omega)$ will be related to Laplace transforms of the autocorrelation. Define

$$S^+(\omega) = \int_0^\infty \mathrm{e}^{\mathrm{i}\omega\tau} C(\tau) \mathrm{d}\tau \tag{67}$$

and

$$S^-(\omega) = \int_{-\infty}^0 \mathrm{e}^{\mathrm{i}\omega\tau} C(\tau) \mathrm{d}\tau = [S^+(\omega)]^* \tag{68}$$

since $C(t) = C(-t)$ by time reversal invariance. Hence

$$S(\omega) = S^+(\omega) + S^-(\omega) = 2 \operatorname{Re} S^+(\omega) \tag{69}$$

if $y(t)$ is a real process.

One may obtain a variety of dispersion relations utilizing, for instance, the fact that $S^+(\omega)$ is analytic in the upper half ω plane, which implies

$$\int_{-\infty}^{\infty} \frac{S^+(\omega')\mathrm{d}\omega'}{\omega' - \omega + \mathrm{i}\varepsilon} = 0, \qquad \varepsilon > 0 \tag{70}$$

and hence that

$$\frac{1}{\pi}\mathrm{P}\int \frac{S^{+}(\omega')\,\mathrm{d}\omega'}{\omega'-\omega} = \mathrm{i}S^{+}(\omega) \tag{71}$$

using the Plemelj formula:

$$\frac{1}{\omega'-\omega\pm\mathrm{i}\varepsilon} = \mathrm{P}\frac{1}{\omega'-\omega} \mp \mathrm{i}\pi\delta(\omega'-\omega). \tag{72}$$

Taking real and imaginary parts, one obtains Kramers–Kronig [see e.g. Landau and Lifshitz (1969)] type relations:

$$\mathrm{Re}\,S^{+}(\omega) = \frac{1}{\pi}\mathrm{P}\int \frac{\mathrm{Im}\,S^{+}(\omega')\,\mathrm{d}\omega'}{\omega'-\omega} \tag{73}$$

$$\mathrm{Im}\,S^{+}(\omega) = -\frac{1}{\pi}\mathrm{P}\int \frac{\mathrm{Re}\,S^{+}(\omega)\,\mathrm{d}\omega}{\omega'-\omega}. \tag{74}$$

The power spectrum and the Laplace transform of the correlation function can thus be constructed from one another and so contain the same information.

The spectral intensity and autocorrelation functions may be generalized, in a uniform background, to include spatial as well as temporal fluctuations:

$$C(\boldsymbol{\rho},\tau) = \langle y(x_0,t_0)\,y(x_0+\boldsymbol{\rho},t_0+\tau)\rangle \tag{75}$$

$$S(\boldsymbol{k},\omega) = \int_V \mathrm{d}^3\rho \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega\tau+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\rho}} C(\boldsymbol{\rho},\tau) \tag{76}$$

and to cross-correlations:

$$C_{ij}(\boldsymbol{\rho},\tau) = \langle y_i(x_0,t_0)\,y_j(x_0+\boldsymbol{\rho},t_0+\tau)\rangle \tag{77}$$

$$S_{ij}(\boldsymbol{k},\omega) = \int_V \mathrm{d}^3\rho \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega\tau+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\rho}} C_{ij}(\boldsymbol{\rho},\tau). \tag{78}$$

In this case time reversal invariance takes the form:

$$C_{ij}(\boldsymbol{\rho},\tau) = \pm C_{ji}(\boldsymbol{\rho},-\tau) \tag{79}$$

$$S_{ij}(\boldsymbol{k},\omega) = \pm S_{ji}^{*}(\boldsymbol{k},\omega). \tag{80}$$

The plus sign arises when the quantities y_i and y_j have the same "parity" under time reversal, the minus sign when they have opposite "parity". Onsager (1931) used arguments based on these time reversal properties to deduce his reciprocity theorem, one of the milestones in the theory of irreversible processes. Under time reversal, magnetic fields reverse direction. The Onsager theorem thus relates transport coefficients in opposite magnetic fields.

The charge density fluctuation spectrum will now be derived. For simplicity, it will be assumed that there is no external magnetic field. The results obtained for a plasma in a uniform external magnetic field are stated without proof later in this section.

The lowest-order theory is obtained by neglecting the Δ term, which is $O(\varepsilon^2)$, in (43), and noting that the electric field of the background plasma vanishes when it is

spatially uniform. Equation (43) then reads:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right)\Gamma_1(X_0,t_0;X,t) - \frac{e_r}{m_r}\frac{\partial f}{\partial \boldsymbol{v}} \cdot\sum_{r'} n_{r'}e_{r'}\int \mathrm{d}X'\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\Gamma_1(X_0,t_0;X',t) = 0 \tag{81}$$

with initial condition given by

$$\Gamma_1(X_0,t_0;X,t_0) = g(X_0,X,t_0)/f(X_0,t_0) + \frac{1}{n_{r_0}}\delta_{rr_0}\delta(X-X_0),$$

which is $O(\varepsilon)$. It should be emphasized again that this evolution equation for Γ is just the linearized Vlasov equation with given initial condition, if g is known. Since the background plasma evolves on a timescale much longer than a fluctuation period, since $\partial f/\partial t = O(\varepsilon)$, Γ_1 may be regarded as a function of a fast variable $t-t_0$ and a slow variable t_0. Laplace transform will be carried out with respect to the fast variable, defining

$$\Gamma_1(\boldsymbol{v}_0;\boldsymbol{v},\boldsymbol{k},\omega) = \int_0^{\infty} \mathrm{e}^{\mathrm{i}\omega(t-t_0)}\mathrm{d}t \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}_0)}\Gamma_1(X_0,t_0;X,t). \tag{82}$$

The power spectrum of charge density fluctuations is then given by:

$$\langle\rho\rho\rangle_{\boldsymbol{k},\omega} = 2\,\mathrm{Re}\langle\rho\rho\rangle^{+}_{\boldsymbol{k},\omega} \tag{83}$$

$$\langle\rho\rho\rangle^{+}_{\boldsymbol{k},\omega} = \sum n_{r_0}e_{r_0}\sum n_r e_r \int \mathrm{d}^3v_0 f(\boldsymbol{v}_0)\int \mathrm{d}^3v\,\Gamma_1(\boldsymbol{v}_0;\boldsymbol{v},\boldsymbol{k},\omega), \tag{84}$$

using (27) and (69). (The ρ here should not be confused with the spatial variable $\boldsymbol{\rho}$ introduced earlier.)

The power spectrum retains a slow dependence on t_0, reflecting an adiabatic variation of the fluctuation spectrum with the changing of the background distribution function f. This can be formalized by the method of multiple timescales, introduced to plasma kinetic theory by Frieman (1963) and Sandri (1963). The validity of the approach requires a clear separation of the timescales of the fluctuations and of the changing background, and so will break down, for instance, whenever the decay time of the fluctuations approaches the collision time.

Fourier transforming (81) gives:

$$(\omega+\boldsymbol{k}\cdot\boldsymbol{v}+\mathrm{i}\varepsilon)\Gamma_1(\boldsymbol{v}_0;\boldsymbol{v},\boldsymbol{k},\omega) - \frac{e_r}{m_r}\frac{4\pi\boldsymbol{k}}{k^2}\cdot\frac{\partial f}{\partial \boldsymbol{v}}\sum_{r'} n_{r'}e_{r'}\int \mathrm{d}^3v'\,\Gamma(\boldsymbol{v}_0;\boldsymbol{v}',\boldsymbol{k},\omega) = \mathrm{i}\left(\frac{g_0(\boldsymbol{v}_0,\boldsymbol{v},\boldsymbol{k})}{f(\boldsymbol{v}_0)} + \delta_{rr_0}\frac{1}{n_r}\delta(\boldsymbol{v}-\boldsymbol{v}_0)\right) \tag{85}$$

ε is a small positive constant (not the plasma parameter!), taken to zero at the end of the calculation, which provides the correct analytic continuation of Γ_1 for real ω.

Dividing (85) through by $\omega + \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}\varepsilon$, integrating over $\boldsymbol{v}$ and summing over species gives:

$$\sum n_r e_r \int \mathrm{d}^3 v\, \Gamma(\boldsymbol{v}_0; \boldsymbol{v}, \boldsymbol{k}, \omega) = \frac{\mathrm{i}}{\varepsilon(\boldsymbol{k},\omega)}\left(\frac{e_{r_0}}{\omega + \boldsymbol{k}\cdot\boldsymbol{v}_0 + \mathrm{i}\varepsilon} + \frac{\sum n_r e_r}{f(\boldsymbol{v}_0)}\int \frac{\mathrm{d}^3 v\, g_0(\boldsymbol{v}_0,\boldsymbol{v},\boldsymbol{k})}{\omega + \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}\varepsilon}\right). \tag{86}$$

$\varepsilon(\boldsymbol{k},\omega)$, the *plasma dielectric function*, is defined by:

$$\varepsilon(\boldsymbol{k},\omega) \equiv 1 - \sum_r \frac{n_r e_r^2}{m_r}\frac{4\pi \boldsymbol{k}}{k^2}\cdot\int \mathrm{d}^3 v \frac{\partial f/\partial \boldsymbol{v}}{\omega + \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}\varepsilon}. \tag{87}$$

Multiplying (86) by $f(\boldsymbol{v}_0)$, integrating over $\boldsymbol{v}_0$ and summing over species gives:

$$\langle\rho\rho\rangle^+_{\boldsymbol{k},\omega} = \sum_{r_0, r} n_{r_0} e_{r_0} n_r e_r \int \mathrm{d}^3 v_0 f(\boldsymbol{v}_0) \int \mathrm{d}^3 v\, \Gamma_1(\boldsymbol{v}_0; \boldsymbol{v}, \boldsymbol{k}, \omega) \tag{88}$$

$$= \frac{\mathrm{i}}{\varepsilon(\boldsymbol{k},\omega)}\sum_r n_r e_r\left(e_r \int \mathrm{d}^3 v \frac{f^r(\boldsymbol{v})}{\omega + \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}\varepsilon} + \int \mathrm{d}^3 v \frac{1}{\omega + \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}\varepsilon}\sum_{r_0} n_{r_0} e_{r_0}\int \mathrm{d}^3 v_0\, g_0(\boldsymbol{v}_0, \boldsymbol{v}, \boldsymbol{k})\right). \tag{89}$$

From this point onward the calculation parallels that of Rostoker (1961). To obtain an explicit result for $\langle\rho\rho\rangle^+_{k,\omega}$, g_0 must be found in terms of f. This is possible, because g changes on the plasma timescale and f changes on the much longer collisional timescale, and thus g is able to follow f "adiabatically". It must be pointed out that the time t_0 is in general not the preparation time of the system, t_p. In general $t_\mathrm{p} \to -\infty$ is taken so that any abnormal correlations have died away. The appropriate g is thus the asymptotic long-time solution of (50). In the context of kinetic theory, this separation of timescales is known as the Bogoliubov (1962) ansatz, and the kinetic equation obtained by substituting the asymptotic g, as a functional of the slowly varying $f(X_0, t_0)$ into (50) for $\partial f/\partial t$ yields the well-known Balescu–Guernsey–Lenard (BGL) equation (109) (Balescu, 1960; Guernsey, 1960; Lenard, 1960). At this point, a critical discussion of the Bogoliubov ansatz would lead us too far astray. Suffice it to say that it breaks down for sufficiently small k (compared with the Debye length), where Landau damping is too weak to damp out the initial correlations sufficiently rapidly, as is required if g is to be validly replaced by its asymptotic long-time limit.

The function g therefore satisfies the following equation, obtained from (50) by dropping the term in h, which is of higher order (it is assumed) in the plasma parameter, and Fourier–Laplace transforming, treating f as constant in time:

$$(\mathrm{i}\varepsilon + \boldsymbol{k}\cdot(\boldsymbol{v}' - \boldsymbol{v}))g(\boldsymbol{v}, \boldsymbol{v}', \boldsymbol{k}) = 4\pi e_r e_{r'}\frac{\boldsymbol{k}}{k^2}\left(\frac{1}{m_{r'}}\frac{\partial f^{r'}}{\partial \boldsymbol{v}'}[f^r(\boldsymbol{v}) + h^r(\boldsymbol{k},\boldsymbol{v})] - \frac{1}{m_r}\frac{\partial f^r}{\partial \boldsymbol{v}}[f^{r'}(\boldsymbol{v}') + h^{r'*}(\boldsymbol{k},\boldsymbol{v}')]\right). \tag{90}$$

Note that g is known once h is known, where $h(\boldsymbol{k},\boldsymbol{v})$ is defined by

$$e_{r_0} h^{r_0}(\boldsymbol{k}, \boldsymbol{v}_0) \equiv \sum_r n_r e_r \int \mathrm{d}^3 v\, g(\boldsymbol{v}_0, \boldsymbol{v}, \boldsymbol{k}). \tag{91}$$

By symmetry under particle interchange

$$e_r h^{r*} = \sum_{r_0} n_{r_0} e_{r_0} \int \mathrm{d}^3 v\, g(\boldsymbol{v}_0, \boldsymbol{v}, \boldsymbol{k}). \tag{92}$$

Make the following convenient definition:

$$U(\boldsymbol{k},\omega) = \frac{k}{\pi}\sum_r n_r e_r^2 \int \mathrm{d}^3 v \frac{f^r(\boldsymbol{v})}{\omega + \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}\varepsilon} \tag{93}$$

and note that the imaginary part of U is given by

$$\operatorname{Im} U(\boldsymbol{k},\omega) = -\sum_r n_r e_r^2 \int \mathrm{d}^3 v\, f^r(\boldsymbol{v})\,\delta\left(\frac{\omega}{k} + \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{k}\right), \tag{94}$$

and that

$$\operatorname{Im}\varepsilon(\boldsymbol{k},\omega) = \frac{\pi}{k}\sum_r 4\pi \frac{n_r e_r^2}{m_r}\frac{\boldsymbol{k}}{k^2}\cdot\int \mathrm{d}^3 v \frac{\partial f^r}{\partial \boldsymbol{v}}\,\delta\left(\frac{\omega}{k} + \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{k}\right). \tag{95}$$

Dividing (90) by $[\boldsymbol{k}\cdot(\boldsymbol{v}' - \boldsymbol{v}) + \mathrm{i}\varepsilon]$, summing over species and integrating gives:

$$h^r(\boldsymbol{v},\boldsymbol{k})\varepsilon(\boldsymbol{k}, -\boldsymbol{k}\cdot\boldsymbol{v}) = (1 - \varepsilon(\boldsymbol{k}, -\boldsymbol{k}\cdot\boldsymbol{v}))f^r(\boldsymbol{v}) - \frac{4\pi}{m_r}\frac{\boldsymbol{k}}{k^2}\cdot\frac{\partial f^r}{\partial \boldsymbol{v}}\left(\frac{\pi}{k}U(\boldsymbol{k}, -\boldsymbol{k}\cdot\boldsymbol{v}) + \int \mathrm{d}^3 v' \sum_{r'} n_{r'} e_{r'}^2 \frac{h^{r'*}(\boldsymbol{k},\boldsymbol{v}')}{\boldsymbol{k}\cdot(\boldsymbol{v}' - \boldsymbol{v} + \mathrm{i}\varepsilon)}\right). \tag{96}$$

Make a further definition:

$$H(u,\boldsymbol{k}) = \sum n_r e_r^2 \int \mathrm{d}^3 v\, h^r(\boldsymbol{v},\boldsymbol{k})\,\delta(u - \boldsymbol{k}\cdot\boldsymbol{v}/k).$$

[Note again that h (hence g) is known once H is known since the right-hand side of (96) depends only on H.] Then, by multiplying (96) by $\delta(\omega + \boldsymbol{k}\cdot\boldsymbol{v})$, integrating over $\boldsymbol{v}$ and summing over species:

$$H(-\omega/k, \boldsymbol{k})\varepsilon(\boldsymbol{k},\omega) = -[1 - \varepsilon(\boldsymbol{k},\omega)]\operatorname{Im} U(\boldsymbol{k},\omega) - \frac{k}{\pi}\operatorname{Im}\varepsilon(\boldsymbol{k},\omega)\left(\frac{\pi}{-k}U(\boldsymbol{k},\omega) + \sum_r n_r e_r^2 \int \mathrm{d}^3 v' \frac{h^*(\boldsymbol{k},\boldsymbol{v}')}{\omega + \boldsymbol{k}\cdot\boldsymbol{v}' + \mathrm{i}\varepsilon}\right). \tag{97}$$

Noting that (89) for $\langle\rho\rho\rangle^+_{k,\omega}$ may be written in terms of U and h with the aid of (92) and (93) as:

$$\langle\rho\rho\rangle^+_{\boldsymbol{k},\omega} = \frac{i}{\varepsilon(\boldsymbol{k},\omega)}\left(\frac{\pi}{k}U(\boldsymbol{k},\omega) + \sum_r n_r e_r^2 \int \mathrm{d}^3 v' \frac{h^*(\boldsymbol{k},\boldsymbol{v}')}{\omega + \boldsymbol{k}\cdot\boldsymbol{v}' + \mathrm{i}\varepsilon}\right), \tag{98}$$

(97) may then be used to express it in terms of H as:

$$\langle\rho\rho\rangle^{+}_{k,\omega}=\frac{\mathrm{i}\pi}{k\,\mathrm{Im}\,\varepsilon(\boldsymbol{k},\omega)}\left[\left(\frac{1}{\varepsilon(\boldsymbol{k},\omega)}-1\right)\mathrm{Im}U(\boldsymbol{k},\omega)-H\left(-\frac{\omega}{k},\boldsymbol{k}\right)\right]. \tag{99}$$

The spectral intensity is obtained by taking twice the real part of the Laplace transform:

$$\langle\rho\rho\rangle_{k,\omega}=\frac{2\pi}{k}\left(-\frac{\mathrm{Im}U(\boldsymbol{k},\omega)}{|\varepsilon(\boldsymbol{k},\omega)|^2}+\frac{\mathrm{Im}\,H(-\omega/k,\boldsymbol{k})}{\mathrm{Im}\,\varepsilon(\boldsymbol{k},\omega)}\right). \tag{100}$$

However, as Lenard (1960) first showed, the imaginary part of H vanishes. This can be seen from the following rewrite of (97) for $H(u)$, by taking its imaginary part using the Plemelj formula (72):

$$H(u)=\int\frac{\mathrm{d}u'}{u'-u-\mathrm{i}\varepsilon}\left[\sum_{r'}\frac{4\pi n_{r'}e_{r'}^2}{m_{r'}k^2}\frac{\partial F^{r'}}{\partial u'}\left(\sum_r n_r e_r^2 F^r(u)+H(u)\right)\right.$$
$$\left.-\sum_r\frac{4\pi n_r e_r^2}{m_r k^2}\frac{\partial F^r}{\partial u}\left(\sum_{r'}n_{r'}e_{r'}^2F^{r'}(u')+H^*(u')\right)\right]. \tag{101}$$

H real does indeed satisfy the above equation and the solution is unique. $F^r(u)$ has been defined by

$$F^r(u)\equiv\int\mathrm{d}^3v\,f^r(\boldsymbol{v})\delta(u-\boldsymbol{k}\cdot\boldsymbol{v}/k). \tag{102}$$

One finally obtains that

$$\langle\rho\rho\rangle_{k,\omega}=\frac{2\pi}{k}\frac{\mathrm{Im}U(\boldsymbol{k},\omega)}{|\varepsilon(\boldsymbol{k},\omega)|^2}\equiv-\frac{2\pi}{k}\frac{\mathrm{Im}U}{\mathrm{Im}\,\varepsilon}\mathrm{Im}(1/\varepsilon) \tag{103}$$

which is a trivial generalization of the result of Rostoker (1961) to a multispecies plasma. Note that the fluctuations grow very large as a mode moves toward marginal stability, but then this theory tends to be inadequate.

In thermal equilibrium the special form of the Maxwellian distribution function enables ε and U to be related:

$$\varepsilon(\boldsymbol{k},\omega)=1+\frac{K^2}{k^2}-\frac{4\pi^3\omega}{Tk^3}U(\boldsymbol{k},\omega), \tag{104}$$

where, as always,

$$K^2\equiv\frac{4\pi}{T}\sum_r n_r e_r^2 \tag{105}$$

is the square of the Debye wavenumber. Hence:

$$\mathrm{Im}U=\frac{-Tk^3}{4\pi^2\omega}\mathrm{Im}\,\varepsilon \tag{106}$$

and

$$\langle\rho\rho\rangle_{k,\omega}=-\frac{Tk^2}{2\pi\omega}\mathrm{Im}\frac{1}{\varepsilon(\boldsymbol{k},\omega)}. \tag{107}$$

Upon use of Poisson's equation relating electric field to charge density,

$$(1/8\pi)\langle \boldsymbol{E}(\boldsymbol{x},t)\boldsymbol{E}(\boldsymbol{x}_0,t_0)\rangle_{k,\omega}=-T\,\mathrm{Im}(1/\omega\varepsilon(\boldsymbol{k},\omega))\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}, \tag{108}$$

which is the result given by applying the fluctuation-dissipation theorem (Callen and Welton, 1951; Callen and Greene, 1952; Kubo, 1957) to the Vlasov plasma [see, e.g., Sitenko (1967)]. [If (103) is compared with (107) and (94) and (95) are used, a fluctuation temperature T_{fl} can be defined by:

$$T_{\mathrm{fl}}\equiv\frac{4\pi^2\omega}{k^3}\frac{\mathrm{Im}U}{\mathrm{Im}\,\varepsilon}=-\frac{\omega}{k}\frac{\sum ne^2\int\mathrm{d}u\,F(u)\delta(u+\omega/k)}{\sum(ne^2/m)\int\mathrm{d}u(\partial F/\partial u)\delta(u+\omega/k)},$$

which shows the competition between Cerenkov emission and Landau damping as determining the spectral intensity. In certain situations, such as due to photo-ionization in the ionosphere, where these are long drawn-out tails in the distribution function, this can be quite high compared with the particle temperature. Of course, in thermal equilibrium they are equal.]

Balescu–Guernsey–Lenard equation

So far obtaining the kinetic equation for $f(X,t)$ has been side-stepped, but all the ingredients for doing so are at hand. From (40), for spatially homogeneous plasma and no external magnetic field,

$$\frac{\partial f^r}{\partial t}(\boldsymbol{v},t)=\frac{e_r}{m_r}\sum_{r'}n_{r'}e_{r'}\int\mathrm{d}X'\frac{\partial}{\partial\boldsymbol{x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\cdot\frac{\partial}{\partial\boldsymbol{v}}g^{rr'}(\boldsymbol{X},\boldsymbol{X}',t), \tag{109}$$

or

$$\frac{\partial f^r}{\partial t}\equiv-\frac{\partial}{\partial\boldsymbol{v}}\cdot\boldsymbol{J}^r, \tag{110}$$

where

$$\boldsymbol{J}^r\equiv-\frac{\mathrm{i}}{(2\pi)^3}\frac{e_r^2}{m_r}\int\mathrm{d}^3k\,\boldsymbol{k}\phi(k)h^r(\boldsymbol{k},\boldsymbol{v}). \tag{111}$$

Here $\phi(k)=4\pi/k^2$, is the Fourier transform of the Coulomb potential $1/|\boldsymbol{x}-\boldsymbol{x}'|$, and $h^r(\boldsymbol{k},\boldsymbol{v})$ is defined by (91). Since $\phi(k)$ is real and the whole r.h.s. of (109) must be real, only the imaginary part of h need be found.

Unfortunately the method of Lenard (1960), who side-stepped solving the integral equation to find Im h, fails in the multi-species case and (96) must be solved by first solving for H from (101). Some notational simplifications will be made. Define

$$\chi(u)\equiv\sum\frac{4\pi ne^2}{mk^2}\frac{\partial F}{\partial u}, \tag{112}$$

and

$$\psi(u)\equiv\sum ne^2F(u). \tag{113}$$

Species labels have been omitted in the sums. Then (101) may be rewritten as:

$$H(u)=[\psi(u)+H(u)]\int\frac{\mathrm{d}u'}{u'-u+\mathrm{i}\varepsilon}\chi(u')$$
$$-\chi(u)\int\frac{\mathrm{d}u'}{u'-u+\mathrm{i}\varepsilon}[\psi(u')+H(u')], \tag{114}$$

remembering H is real.

This is further simplified, writing

$$K(u)=+\psi(u)+K(u)\int\frac{\mathrm{d}u'\chi(u')}{u'-u+\mathrm{i}\varepsilon}-\chi(u)\int\mathrm{d}u'\frac{K(u')}{u'-u+\mathrm{i}\varepsilon}, \tag{115}$$

where

$$K(u)\equiv H+\psi, \tag{116}$$

Define

$$A(z)\equiv\int\mathrm{d}u\frac{K(u)}{u-z},\qquad A^{\pm}(u)=\int\mathrm{d}u'\frac{K(u')}{u'-u\mp\mathrm{i}\varepsilon},$$
$$B(z)\equiv\int\mathrm{d}u\frac{\chi(u)}{u-z},\qquad B^{\pm}(u)\equiv\int\mathrm{d}u'\frac{\chi(u')}{u'-u\mp\mathrm{i}\varepsilon}.$$

Now the functions $A(z)$ and $B(z)$ are analytic everywhere in the complex z plane except along the real axis where they have jumps $2\pi\mathrm{i}K(u)$ and $2\pi\mathrm{i}\chi(u)$, respectively. (These functions K and χ are real functions!)

Now rewrite (114) as

$$\frac{1}{2\pi\mathrm{i}}[A^+(u)-A^-(u)]=\psi(u)+\frac{1}{2\pi\mathrm{i}}(A^+-A^-)B^- -\frac{1}{2\pi\mathrm{i}}(B^+-B^-)A^-$$

or

$$\frac{1}{2\pi\mathrm{i}}[A^+(1-B^-)-A^-(1-B^+)]=\psi(u)$$

If this equation is divided by $(1-B^-)(1-B^+)$,

$$\left(\frac{A^+}{1-B^+}-\frac{A^-}{1-B^-}\right)=\frac{2\pi\mathrm{i}\psi(u)}{(1-B^+)(1-B^-)}.$$

Notice $(1-B^-)$ is just the plasma dispersion function $\varepsilon(k,-\mathrm{i}ku)$ and for stable plasmas has no zeros in the lower-half z plane. Likewise $(1-B^+)=(1-B^-)^*$ has no zeros in the opposite half-plane: the denominators do not vanish. A generalization of Cauchy's Theorem is now used to write the solution,

$$\frac{A(z)}{1-B(z)}=\int\mathrm{d}u'\frac{\psi(u')/|\varepsilon(k,-ku')|^2}{u-z}. \tag{117}$$

Hence

$$A(z)=[1-B(z)]\int\mathrm{d}u'\frac{\psi(u')/|\varepsilon(k,-ku')|^2}{u'-z}. \tag{118}$$

It can be seen from (96) what is needed for solving for h is just A^-. Hence

$$A^-=\varepsilon(k,-k\cdot v)\int\mathrm{d}u'\frac{\psi(u')/|\varepsilon|^2}{u'-u+\mathrm{i}\varepsilon}. \tag{119}$$

Now the equation for h^r may be rewritten as

$$h^r(v)\varepsilon(k,-k\cdot v)=f^r(v)(1-\varepsilon)-\frac{4\pi}{k^2m_r}\frac{\partial f^r(v)}{\partial u}A^-(u). \tag{120}$$

Using the result for A^-,

$$h^r(v)=f^{(r)}(v)\frac{(1-\varepsilon)}{\varepsilon}-\frac{4\pi}{k^2m_r}\frac{\partial f^r(v)}{\partial u}\int\mathrm{d}u'\frac{\psi(u')/|\varepsilon(k,-ku')|^2}{u'-u+\mathrm{i}\varepsilon}. \tag{121}$$

Upon taking the imaginary part,

$$\operatorname{Im}h^r(v)=f^r(v)\frac{\operatorname{Im}\varepsilon^*}{|\varepsilon|^2}+\frac{4\pi^2}{k^2m_r}\frac{\partial f^r(v)}{\partial u}\frac{\sum ne^2F}{|\varepsilon|^2}$$
$$=-\frac{\pi f^r(v)}{|\varepsilon|^2}\sum\frac{4\pi ne^2}{m}\frac{\partial F(u)}{\partial u}+\frac{4\pi^2}{k^2m_r}\frac{\partial f^r(v)}{\partial u}\sum\frac{ne^2F(u)}{|\varepsilon|^2}. \tag{122}$$

Thus the expression for J^r may be written down:

$$J^r(v)=-2\frac{e_r^2}{m_r}\int\mathrm{d}^3v'\int\mathrm{d}^3k$$
$$\cdot\frac{kk\cdot[(1/m_r)(\partial f^r/\partial v)\sum ne^2f(v')-f^r(v)\sum(ne^2/m)(\partial f(v')/\partial v')]}{k^4|\varepsilon(k,-k\cdot v)|^2}$$
$$\times\delta(k\cdot v-k\cdot v'), \tag{123}$$

where the terms in the sums Σ are over all species.

These kinetic equations [following Lenard (1960)] may be shown to retain their positivity if initially positive, to yield particle conservation, and, when summed over species, to be momentum and energy conserving. Further the Maxwellian distribution is a stationary solution and an H-theorem may be demonstrated.

Inclusion of uniform magnetic field

In other parts of this section the important effects of a uniform magnetic field have not been included. To show how to include the effects of such a field the dispersion relation for electrostatic oscillations will be derived using a technique due to Rostoker (1961). To include the field in equations for correlation functions, etc., see the paper of Oberman and Shure (1963), for example.

Consider perturbations $f_1(x,v,t)$, $E_1(x,t)=-\nabla\psi$ about an equilibrium $f_0(v_\perp,v_\parallel)$, $\partial f_0/\partial\phi=0$, where v is given by

$$v\equiv(v_\perp\cos\phi,v_\perp\sin\phi,v_\parallel)\equiv e_\perp v_\perp+e_\parallel v_\parallel$$

in cylindrical coordinates in velocity space, with $v_\parallel\equiv v\cdot B/B$.

For perturbations of the species s of the form

$$f_1^s\propto e^{\mathrm{i}k\cdot x+\mathrm{i}\omega t},\qquad \gamma=\operatorname{Im}\omega<0\text{ for instability},$$

then since

$$\frac{\partial}{\partial \boldsymbol{v}} = \boldsymbol{e}_\perp \frac{\partial}{\partial v_\perp} + \frac{\boldsymbol{e}_\phi}{v_\perp}\frac{\partial}{\partial \phi} + \boldsymbol{e}_\parallel \frac{\partial}{\partial v_\parallel},$$

$$(+\mathrm{i}\omega + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}) f_1^s - \Omega_s \frac{\partial f_0^s}{\partial \phi} = \frac{\mathrm{i} e_s}{m_s}\psi\left(\boldsymbol{k}\cdot\boldsymbol{e}_\perp \frac{\partial f_0^s}{\partial v_\perp} + k_\parallel \frac{\partial f_0^s}{\partial v_\parallel}\right) \tag{124}$$

or

$$(+\mathrm{i}\omega + \mathrm{i}k_\parallel v_\parallel) f_1^s + \mathrm{i}k_\perp v_\perp \cos(\phi-\alpha) f_1^s - \Omega_s \frac{\partial f_0^s}{\partial \phi}$$
$$= \frac{\mathrm{i} e_s}{m_s}\psi\left(k_\perp \cos(\phi-\alpha)\frac{\partial f_0^s}{\partial \phi} + k_\parallel \frac{\partial f_0^s}{\partial v_\parallel}\right). \tag{125}$$

Here $\Omega_s = e_s B/m_s c$ and $k = (k_\perp \cos\alpha, k_\perp \sin\alpha, k_\parallel)$.

Now let any function $A(\boldsymbol{k}, \boldsymbol{v})$ have the expansion

$$A(\boldsymbol{k},\boldsymbol{v}) = \exp[\mathrm{i}k_\perp a_s \sin(\phi-\alpha)] \times \sum_{n=-\infty}^{\infty} J_n(k_\perp a_s)\exp[-\mathrm{i}n(\phi-\alpha)] A_n(\boldsymbol{k}, v_\perp, v_\parallel) \tag{126}$$

with the inversion

$$A_n(\boldsymbol{k}, v_\perp, v_\parallel) = \frac{1}{2\pi J_n(k_\perp a_s)} \times \int_0^{2\pi} \mathrm{d}\phi\, A(\boldsymbol{k},\boldsymbol{v})\exp[-\mathrm{i}k_\perp a_s \sin(\phi-\alpha)]\exp[\mathrm{i}n(\phi-\alpha)]. \tag{127}$$

Here $a_s = v_\perp/\Omega_s$ and the J_n are ordinary Bessel functions of the first kind. Repeated use is made of the identity

$$\exp[\mathrm{i}k_\perp a_s \sin(\phi-\alpha)] = \sum_{n=-\infty}^{\infty} J_n(k_\perp a_s)\exp[\mathrm{i}n(\phi-\alpha)]. \tag{128}$$

[Note if $\partial A(\boldsymbol{k},\boldsymbol{v})/\partial\phi = 0$ then it follows at once that $A(\boldsymbol{k},\boldsymbol{v}) = A_n(\boldsymbol{k}, v_\perp, v_\parallel)$.] Now multiply (125) by

$$\frac{1}{2\pi J_n}\exp[\mathrm{i}k_\perp a_s \sin(\phi-\alpha)]\exp[(\mathrm{i}n(\phi-\alpha)]$$

and integrate over ϕ. There results

$$(+\mathrm{i}\omega + \mathrm{i}k_\parallel v_\parallel) f_n^s(\boldsymbol{k}, v_\perp, v_\parallel)$$
$$- \frac{\Omega_s}{2\pi J_n}\int_0^{2\pi} \mathrm{d}\phi \frac{\partial}{\partial\phi}\{\exp[-\mathrm{i}k_\perp a_s \sin(\phi-\alpha)]\}\exp[\mathrm{i}n(\phi-\alpha)] f^s$$
$$- \frac{\Omega_s}{2\pi J_n}\int_0^{2\pi} \mathrm{d}\phi\, \exp[-\mathrm{i}k_\perp a_s \sin(\phi-\alpha)]\frac{\partial f^s}{\partial\phi}\exp[\mathrm{i}n(\phi-\alpha)]$$
$$= \frac{\mathrm{i}\psi e_s}{m_s}\frac{1}{2\pi J_n}\int \mathrm{d}\phi \exp[-\mathrm{i}k_\perp a_s \sin(\phi-\alpha)]\mathrm{e}^{\mathrm{i}n\phi}\left(k_\parallel \frac{\partial f_0^s}{\partial v_\parallel} + k_\perp \cos(\phi-\alpha)\frac{\partial f_0^s}{\partial v_\perp}\right).$$

It follows readily upon integration by parts that

$$(+\mathrm{i}\omega + \mathrm{i}k_\parallel v_\parallel + \mathrm{i}n\Omega_s) f_n = \frac{\mathrm{i}e_s}{m_s}\psi\left(k_\parallel \frac{\partial f_0^s}{\partial v_\parallel} + \frac{k_\perp}{J_n}\frac{(J_{n+1}+J_{n-1})}{2}\frac{\partial f_0^s}{\partial v_\perp}\right)$$
$$= \frac{\mathrm{i}e_s}{m_s}\psi\left(k_\parallel \frac{\partial f_0^s}{\partial v_\parallel} + \frac{n}{a_s}\frac{\partial f_0^s}{\partial v_\perp}\right),$$

where the identity

$$J_{n+1}(z) + J_{n-1}(z) = (2n/z)J_n(z)$$

has been used. Therefore,

$$f_n = \frac{e_s}{m_s}\psi\frac{k_\parallel \partial f_0^s/\partial v_\parallel + (n/a_s)\partial f_0^s/\partial v_\perp}{k_\parallel v_\parallel + n\Omega_s + \omega}.$$

Now from (126) and (127) it follows that

$$\int \mathrm{d}^3 v\, A(\boldsymbol{k},\boldsymbol{v}) = \int \mathrm{d}^3 v \sum_{n=-\infty}^{\infty} J_n^2 A_n.$$

Hence

$$\psi = \frac{4\pi}{k^2}\sum_s e_s \int f_s \mathrm{d}^3 v$$
$$= \psi \sum_s\sum_n \frac{\omega_{ps}^2}{k^2}\int \mathrm{d}^3 v\, J_n^2(k_\perp a_s)\frac{k_\parallel \partial f_0^s/\partial v_\parallel + (n/a_s)\partial f_0^s/\partial v_\perp}{k_\parallel v_\parallel + n\Omega_s + \omega}.$$

Hence

$$\left(1 - \sum_s\sum_n \frac{\omega_{ps}^2}{k^2}\int \mathrm{d}^3 v\, J_n^2(k_\perp a_s)\frac{k_\parallel \partial f_0^s/\partial v_\parallel + (n/a_s)\partial f_0^s/\partial v_\perp}{k_\parallel v_\parallel + n\Omega_s + \omega}\right)\psi$$
$$= \varepsilon(\boldsymbol{k},\omega)\psi = 0 \tag{129}$$

yields the dispersion relation. For stable plasmas the analytic contuation for $\operatorname{Im}\omega \geqslant 0$ is effected by deforming the $v_\parallel$ integration from the real line into the Landau contour, just as in the absence of magnetic field.

The equations for the spectral intensity found earlier in this section may be readily obtained using this technique. $\varepsilon(\boldsymbol{k},\omega)$ and $U(\boldsymbol{k},\omega)$ are just replaced by

$$\varepsilon(\boldsymbol{k},\omega) = 1 - \sum_s \frac{\omega_p^2}{k^2}\int \mathrm{d}^3 v \sum_n \frac{J_n^2(ka_\perp)[\boldsymbol{k}\cdot\partial f/\partial \boldsymbol{v}]_n}{\omega + (\boldsymbol{k}\cdot\boldsymbol{v})_n + \mathrm{i}\varepsilon}$$
$$U(\boldsymbol{k},\omega) = \frac{k}{\pi}\sum ne^2 \int \mathrm{d}^3 v\, f(v)\sum_n \frac{J_n^2}{\omega + (\boldsymbol{k}\cdot\boldsymbol{v})_n + \mathrm{i}\varepsilon}. \tag{130}$$

Here

$$\boldsymbol{B}_0 \equiv B_0\hat{z}, \qquad \boldsymbol{k} = k_z\hat{z} + \boldsymbol{k}_\perp,$$

and

$$(\boldsymbol{k}\cdot\boldsymbol{v})_n \equiv k_z v_z + n\Omega_s,$$

$$\left(\boldsymbol{k}\cdot\frac{\partial f}{\partial \boldsymbol{v}}\right)_n \equiv k_z\frac{\partial f}{\partial v_z} + \frac{n}{a}\frac{\partial f}{\partial v}.$$

Scattering from density fluctuations

From the Lienard–Wiechert potentials [See e.g. Jackson (1962)] expressions may be obtained for the electromagnetic fields at a point $\boldsymbol{r}$ due to a moving charge at $\boldsymbol{\rho}(t)$:

$$\boldsymbol{E}(\boldsymbol{r},t) = e\left(\frac{(1-\beta^2)(\boldsymbol{N}-\boldsymbol{B})}{g^3R^2} + \frac{\boldsymbol{N}\times[(\boldsymbol{N}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{g^3cR}\right)_{t'}, \tag{131}$$

$$\boldsymbol{B} = \boldsymbol{N}\times\boldsymbol{E},$$

where $\boldsymbol{\beta} \equiv \boldsymbol{v}/c$, $g = (1-\boldsymbol{N}\cdot\boldsymbol{\beta})$, $R \equiv |\boldsymbol{r}-\boldsymbol{\rho}(t)|$, and $\boldsymbol{N}$ is a unit vector along $\boldsymbol{R}$. The subscript t' means all quantities are to be evaluated at the time $t' = t - R(t')/c$. In the wave-zone the second term in (131) dominates and in the nonrelativistic limit this becomes

$$\boldsymbol{E}(\boldsymbol{r},t) = (e/Rc^2)[\boldsymbol{N}\times(\boldsymbol{N}\times\dot{\boldsymbol{v}})]_{t'} \tag{132}$$

Envisage a large volume of plasma, on the average spatially uniform, illuminated by a plane wave $\boldsymbol{E}_0\cos(\boldsymbol{K}\cdot\boldsymbol{x}-\Omega t)$. This induces an acceleration to each charge given by

$$\dot{\boldsymbol{v}}_i = (e/m)\boldsymbol{E}_0\cos(\boldsymbol{K}\cdot\boldsymbol{\rho}_i(t)-\Omega t),$$

in the first Born approximation (i.e, it is assumed that the plasma is optically thin). Summing the contribution from all particles and neglecting the contribution from the ions, which is down by the mass ratio, gives:

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{e^2}{mc^2}\int \mathrm{d}^3\rho\int \mathrm{d}t'\delta\frac{[t'-t+|\boldsymbol{r}-\boldsymbol{\rho}(t')|/c]}{|\boldsymbol{r}-\boldsymbol{\rho}(t')|}$$
$$\times\boldsymbol{N}(t')\times[\boldsymbol{N}(t')\times\boldsymbol{E}_0]\cos[\boldsymbol{K}\cdot\boldsymbol{\rho}(t')-\Omega t']\sum_j\delta[\boldsymbol{\rho}-\boldsymbol{\rho}_j(t')]. \tag{133}$$

At large distances from the plasma ($|\boldsymbol{r}| \gg |\boldsymbol{\rho}|$), representing the δ-function in ω-space,

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{r_0}{R}[\boldsymbol{E}_0 - \boldsymbol{N}\boldsymbol{N}\cdot\boldsymbol{E}_0]\mathrm{Re}\exp[-\mathrm{i}\omega(t-R/c)]$$
$$\times\int\frac{\mathrm{d}\omega}{2\pi}\int \mathrm{d}^3\rho\int \mathrm{d}t'\exp\left\{\mathrm{i}\left[(\omega-\Omega)t' + \left(\boldsymbol{K}-\frac{\omega}{c}\boldsymbol{N}\right)\cdot\boldsymbol{\rho}\right]\right\}\sum_j\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_j(t')). \tag{134}$$

Here r_0 is the classical electron radius e^2/mc^2. The quantity of interest is the expectation value

$$\lim_{T\to\infty}\frac{1}{T}\int \mathrm{d}t\langle|E^2|\rangle$$
$$= \int\frac{\mathrm{d}\omega}{2\pi}N\frac{r_0^2}{R^2}|E_0|^2[1-\sin^2\alpha\cos^2(\phi-\phi_0)]S\left(\boldsymbol{K}-\frac{\omega}{c}N, \omega-\Omega\right). \tag{135}$$

Here α is the angle between $\boldsymbol{N}$ and $\boldsymbol{K}$, and ϕ and ϕ_0 are azimuthal angles locating $\boldsymbol{N}$ and $\boldsymbol{E}_0$ in a plane perpendicular to $\boldsymbol{K}$ respectively. If the incident wave is unpolarized the factor $\cos^2(\phi-\phi_0)\to 1/2$. The quantity

$$S(\boldsymbol{k},\omega) = \lim_{V,T\to\infty}\frac{2}{VT}[\langle|n_e(\mathrm{k},\omega)|^2/n\rangle] \tag{136}$$

is the spectral power density of electron density fluctuations. Hence, the time-averaged mean energy scattered per unit time per unit solid angle Ω, per unit $\Delta\omega/2\pi$ (a band pass filter is placed at $\boldsymbol{r}$ permitting only frequencies in the interval $\Delta\omega$)

$$\frac{\mathrm{d}W}{\mathrm{d}t\,\mathrm{d}\Omega\,\Delta\omega/2\pi} = nVS(\boldsymbol{k},\omega)r_0^2[1-\sin^2\alpha\sin^2(\phi-\phi_0)]. \tag{137}$$

The quantity S may be computed using (27) and (82), and parallels exactly the calculation for the spectral intensity for charge density fluctuations given earlier. Only the results will be quoted:

$$\frac{1}{2\pi}S(\boldsymbol{k},\omega) = \frac{|(1+\chi_i)|^2}{|\varepsilon|^2}\int \mathrm{d}^3v f_e(\boldsymbol{v})\delta(\omega+\boldsymbol{k}\cdot\boldsymbol{v}) + \frac{|\chi_e|^2}{|\varepsilon|^2}\int \mathrm{d}^3v f_i(\boldsymbol{v})\delta(\omega+\boldsymbol{k}\cdot\boldsymbol{v}). \tag{138}$$

Here the χ's are defined through

$$\varepsilon(\boldsymbol{k},\omega) \equiv 1 + \chi_e(\boldsymbol{k},\omega) + \chi_i(\boldsymbol{k},\omega). \tag{139}$$

A detailed discussion of these results for different situations, as well as references, are given in the paper by Rosenbluth and Rostoker (1962). [See also Bekefi (1966).] Only a few properties will be quoted:

(a) The scattering is only from electrons, i.e. from electrons in the polarization cloud of moving ions and from the core and the electronic part of the cloud of moving electrons.

(b) If $\omega \sim \Omega$ then the scattering is from particles near zero velocity. Since the equation for the spectral intensity depends only on $F_e(u\cong 0) = (1/2\pi)^{1/2}/v_{\mathrm{Te}}$ [see (102)] and $F_i(u\cong 0) = (1/2\pi)^{1/2}/v_{\mathrm{Ti}}$, and for approximately equal temperature $v_{\mathrm{Ti}} \ll v_{\mathrm{Te}}$, the plot of the scattering cross section against $(\omega-\Omega)$ should have Doppler broadening $\Delta\omega \sim kv_{\mathrm{Ti}}$, and indeed this has been observed for $|\boldsymbol{k}\lambda_{\mathrm{D}}| \ll 1$ [see Evans and Loewenthal (1964)].

(c) When $(\omega-\Omega) \sim \omega_{\mathrm{pe}}$ there should be a sharp resonance, for $|\boldsymbol{k}\lambda_0|$ small, and indeed this has been seen in the Arecibo radar backscattering experiments (Perkins et al., 1965). This occurs, of course, because of the resonance at $\omega \sim \omega_{\mathrm{pe}}$ in $\varepsilon(\boldsymbol{k},\omega)$.

(d) If $T_e \gg T_i$, $|\boldsymbol{k}\lambda_0| \ll 1$, then acoustic waves become resonances of $\varepsilon(\boldsymbol{k},\omega)$ and "horns" should be observed at the acoustic frequency.

(e) If $|\boldsymbol{k}\lambda_{\mathrm{D}}| \gg 1$, the plasma cannot support collective behavior (except at nearly backscatter) and then

$$S(\boldsymbol{k},\omega) \cong 2\pi\int \mathrm{d}^3v f_e(\boldsymbol{v})\delta(\omega+\boldsymbol{k}\cdot\boldsymbol{v}), \tag{140}$$

i.e. just the formula for scattering from unscreened electrons and indeed the electron distribution can be probed (using lasers), and if Maxwellian, the temperature determined.

(f) In thermal equilibrium the scattering cross section integrated over all frequencies can be computed to yield

$$\int \frac{\mathrm{d}\omega}{2\pi} S(\boldsymbol{k},\omega) = \frac{1+(k\lambda_{\mathrm{De}})^2}{2+(k\lambda_{\mathrm{De}})^2} \simeq \begin{cases} \frac{1}{2} & |k\lambda_{\mathrm{De}}| \ll 1 \\ 1 & |k\lambda_{\mathrm{De}}| \gg 1. \end{cases} \tag{141}$$

The plasma polarization effects for $|k\lambda_{\mathrm{D}}|$ small have reduced the usual Thomson scattering cross section by 1/2.

Enhanced induced emission — Rostoker superposition principle

A quite general calculation will be presented here for the induced emission of high-frequency waves in plasma when the amplitude of the incident wave is just subcritical for the inducement of parametric instabilities. This enhanced emission, which can be well above the thermal level in the absence of the incident wave, has been observed in ionospheric and laboratory (and computer) experiments. (Actually, the enhanced emission persists post critical, but the intensity of excited waves is then determined primarily by the competition between instability and nonlinear saturation [see e.g. Valeo et al. (1972)]. The calculation begins with the unified formalism of Drake et al. (1974) for treating parametric interactions, and then invokes the superposition principle of Rostoker (1964) [see also Krommes (1976)] of dressed then uncorrelated particles for introducing particle discreteness.

Consider a large amplitude plane-polarized electromagnetic wave

$$\begin{aligned} \boldsymbol{E} &\equiv 2\boldsymbol{e}_0 E_0 \cos(\boldsymbol{K}\cdot\boldsymbol{x} - \Omega t) \\ &\equiv \boldsymbol{E}_{0+}\exp[\mathrm{i}(\boldsymbol{K}\cdot\boldsymbol{x} - \Omega t)] + \boldsymbol{E}_{0-}\exp[-\mathrm{i}(\boldsymbol{K}\cdot\boldsymbol{x} - \Omega t)] \end{aligned} \tag{142}$$

propagating in a spatially homogeneous plasma. [The notation of Drake et al. will be used for purposes of symmetry. Note that the Fourier transform convention is different from that previously employed. (The factor 2 in (142) spares propagating through the calculation a factor of 1/2)]. The coupling of the pump to electron fluctuations $\delta n_{\mathrm{e}}(\boldsymbol{k},\omega)$ arising from the discrete charged particles and their clouds lead to sidebands at $\boldsymbol{k}_{\pm l} \equiv \boldsymbol{k} \pm l\boldsymbol{K}$, $\omega_{\pm l} = \omega \pm l\Omega$. Since situations are considered where $eE_0/m\Omega c \ll 1$, only the first side bands at $\boldsymbol{k}_{\pm} \equiv \boldsymbol{k} \pm \boldsymbol{K}$, $\omega_{\pm} \equiv \omega \pm \Omega$ will be of concern.

The Fourier transform of the wave equation for the side band $E_{\pm}(\boldsymbol{k}_{\pm},\omega_{\pm})$ may be written

$$\left[\left(k_{\pm}^2 - \omega_{\pm}^2/c^2\right)\boldsymbol{I} - \boldsymbol{k}_{\pm}\boldsymbol{k}_{\pm}\right]\cdot\boldsymbol{E}_{\pm} = \left(4\pi\mathrm{i}\omega_{\pm}/c^2\right)\boldsymbol{J}_{\pm}. \tag{143}$$

Here $\boldsymbol{I}$ is the unit dyadic. The current density consists of the sum of two parts, the usual linear response given by $\boldsymbol{\sigma}_{\pm}\cdot\boldsymbol{E}_{\pm}$, and the part

$$\delta\boldsymbol{J}_{\pm} = \pm\frac{\mathrm{i}e^2}{m\Omega}\delta n_{\mathrm{e}}(\boldsymbol{k},\omega)\boldsymbol{E}_{0\pm}, \tag{144}$$

produced by the beating of the high-frequency oscillation velocity induced by the

pump and the low-frequency electrostatic density fluctuation. In what follows $\Omega/\omega_{\mathrm{p}} \gg 1$ (or if $\Omega/\omega_{\mathrm{p}} \sim 1$, then $k\lambda_{\mathrm{D}} \ll 1$) and $\omega/\Omega \ll 1$ are taken.

Equation (143) is now written as:

$$\begin{aligned} &\left[\left[\left(k_{\pm}^2 c^2 - \omega_{\pm}^2\right) - 4\pi\mathrm{i}\sigma_{\perp}\omega\right]\left(\boldsymbol{I} - \hat{\boldsymbol{k}}_{\pm}\hat{\boldsymbol{k}}_{\pm}\right) - \omega^2\left(1 + \frac{4\pi\mathrm{i}\sigma_{\parallel}}{\omega}\right)\hat{\boldsymbol{k}}_{\pm}\hat{\boldsymbol{k}}_{\pm}\right]\boldsymbol{E}_{\pm} \\ &\quad = -\omega_{\mathrm{p}}^2\frac{\delta n_{\mathrm{e}}(\boldsymbol{k},\omega)}{n_0}\boldsymbol{E}_{0\pm}. \end{aligned} \tag{145}$$

Here, $\hat{\boldsymbol{k}} \equiv \boldsymbol{k}/|k|$.

For isotropic velocity distributions,

$$\boldsymbol{\sigma}_{+} = \sigma_{\perp\pm}\left(\boldsymbol{I} - \hat{\boldsymbol{k}}_{\pm}\hat{\boldsymbol{k}}_{\pm}\right) + \sigma_{\parallel\pm}\hat{\boldsymbol{k}}_{\pm}\hat{\boldsymbol{k}}_{\pm}, \tag{146}$$

with

$$\sigma_{\perp\pm} = \frac{\omega_{\mathrm{p}}^2}{4\pi\mathrm{i}k_{\pm}}\int \mathrm{d}u\,\frac{F_0(u)}{u - \omega_{\pm}/k_{\pm} - \mathrm{i}\nu/k_{\pm}},$$

and

$$\sigma_{\parallel\pm} = -\frac{\omega_{\mathrm{p}}^2}{4\pi\mathrm{i}k_{\pm}}\int \mathrm{d}u\,\frac{u\,\partial F_0(u)/\partial u}{u - \omega_{\pm}/k_{\pm} - \mathrm{i}\nu/k_{\pm}},$$

and

$$F_0(u) \equiv \int \mathrm{d}^3v\, f_0(\boldsymbol{v})\,\delta\left(u - \hat{\boldsymbol{k}}_{\pm}\cdot\boldsymbol{v}\right)$$

has been defined earlier.

Write (145) as

$$\left(D_{\pm}\left(\boldsymbol{I} - \hat{\boldsymbol{k}}_{\pm}\hat{\boldsymbol{k}}_{\pm}\right) - \omega_{\pm}^2\varepsilon_{\pm}\hat{\boldsymbol{k}}_{\pm}\hat{\boldsymbol{k}}_{\pm}\right)\cdot\boldsymbol{E}_{\pm} = -\omega_{\mathrm{pe}}^2\frac{\delta n_{\mathrm{e}}}{n_0}(\boldsymbol{k},\omega)\boldsymbol{E}_{0\pm}, \tag{147}$$

with $D_{\pm}$ and $\varepsilon_{\pm}$ well approximated for high frequencies by

$$D_{\pm} \doteq k_{\pm}^2 c^2 + \omega_{\mathrm{p}}^2 - \omega_{\pm}^2 - \mathrm{i}\omega_{\mathrm{p}}^2\nu/\omega_{\pm}, \tag{148}$$

and

$$\varepsilon_{\pm} \doteq 1 - \left(\omega_{\mathrm{p}}^2/\omega_{\pm}^2\right)\left(1 + 3k_{\pm}^2 v_{\mathrm{Te}}^2\right) + \left(\mathrm{i}\omega_{\mathrm{p}}^2/\omega_{\pm}^2\right)\nu/\omega_{\pm}. \tag{149}$$

Here ν is the electron–ion collision frequency [or possibly Landau damping in (149), depending upon the size of $k_{\pm}\lambda_{\mathrm{D}}$]. Equation (147) is readily inverted to yield

$$\boldsymbol{E}_{\pm} = -\omega_{\mathrm{p}}^2\frac{\delta n_{\mathrm{e}}(\boldsymbol{k},\omega)}{n_0}\left(\frac{\left(\boldsymbol{I} - \hat{\boldsymbol{k}}_{\pm}\hat{\boldsymbol{k}}_{\pm}\right)}{D_{\pm}} - \frac{\hat{\boldsymbol{k}}_{\pm}\hat{\boldsymbol{k}}_{\pm}}{\omega_{\pm}^2\varepsilon_{\pm}}\right)\cdot\boldsymbol{E}_{0\pm}.$$

The next task is to compute $\delta n_{\mathrm{e}}(\boldsymbol{k},\omega)$. It is computed from the set of equations:

$$\delta n_{\mathrm{e}}(\boldsymbol{k},\omega) = \frac{\chi_{\mathrm{e}}}{4\pi e}(\boldsymbol{k},\omega)\mathrm{i}\boldsymbol{k}\cdot\left(\boldsymbol{E}(\boldsymbol{k},\omega) + \frac{\mathrm{i}\boldsymbol{k}}{e}\psi(\boldsymbol{k},\omega)\right) + S_{\mathrm{e}}(\boldsymbol{k},\omega), \tag{150}$$

$$\delta n_{\mathrm{i}}(\boldsymbol{k},\omega) = -\frac{\chi_{\mathrm{i}}(\boldsymbol{k},\omega)}{4\pi e}\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{E}(\boldsymbol{k},\omega) + S_{\mathrm{i}}(\boldsymbol{k},\omega), \tag{151}$$

and

$$ik\cdot E(k,\omega)=4\pi e(\delta n_i-\delta n_e). \tag{152}$$

These equations represent the collective (Vlasov) response of the plasma, in the presence of the pump, to the bare fluctuating microdensity. That is, e.g.,

$$S_i(k,\omega)=\frac{1}{(2\pi)^3}\sum_i \exp(-ik\cdot x_{0i})\delta(\omega-k\cdot v_i), \tag{153}$$

is the Fourier transform of

$$S(x,t)=\sum_i \delta(x-x_{0i}-v_i t), \tag{154}$$

the bare ion microdensity. A similar equation holds for the electron S_e. The χ's are the linear Vlasov susceptibilities. The ponderomotive potential which represents the influence of the pump, enters in the electron Vlasov response as a force on the electrons

$$F_\omega=-\nabla\psi_\omega$$

with

$$\psi_\omega=\frac{e^2}{2m}\left\langle\left(\mathrm{Re}\left|\frac{E_{0+}}{i\Omega}+\frac{E_+}{i\omega_+}+\frac{E_-}{i\omega_-}\right|\right)^2\right\rangle_\omega^T$$

$$\cong(e^2/m\Omega^2)(E_{0+}\cdot E_-+E_{0-}\cdot E_+). \tag{155}$$

The brackets $\langle\rangle_\omega^T$ represents the ω frequency component remaining after a time average over the fast timescale characterized by $\Omega\gg\omega$. A corresponding term is not retained in the ion equation because it would be smaller in the electron–ion mass ratio compared with the terms retained. The derivation of the equations, without the sources, is discussed in Drake et al. There the fluctuations were somehow initially imposed: if above threshold they would grow, if not, they would die away. Here the subcritical situation sustained by the microscopic sources is considered.

Equations (150)–(152) may be solved for δn_e:

$$\delta n_e=-\frac{1}{\varepsilon(k,\omega)}\left(\chi_e(k,\omega)S_i+(1+\chi_i)S_e-\frac{k^2\psi}{4\pi e^2}\chi_e(1+\chi_i)\right) \tag{156}$$

where $\varepsilon(k,\omega)\equiv 1+\chi_e+\chi_i$ is the low-frequency dispersion relation. Equation (156) may now be substituted into the equation for $E_\pm$ to obtain

$$E_\pm=-\frac{\omega_p^2}{n_0}\frac{1}{\varepsilon_{NL}(k,\omega)}[\chi_e(k,\omega)S_i(k,\omega)+(1+\chi_i)S_e]\left(\frac{I_{\perp\pm}}{D_\pm}-\frac{\hat{k}_\pm\hat{k}_\pm}{\omega_\pm^2\varepsilon_\pm}\right)\cdot E_{0\pm}, \tag{157}$$

where ε_{NL} is the nonlinear dispersion relation

$$\varepsilon_{NL}=\varepsilon(k,\omega)-k^2\chi_e(1+\chi_i)\left(\frac{|\hat{k}_+\times v_0|^2}{D_+}+\frac{|\hat{k}_-\times v_0|^2}{D_-}-\frac{|\hat{k}_+\cdot v_0|^2}{\omega_+^2\varepsilon_+}-\frac{|\hat{k}_-\cdot v_0|^2}{\omega_-^2\varepsilon_-}\right) \tag{158}$$

with $v_0=eE_0/m\Omega$. The near vanishing of ε_{NL} is the critical condition. The expectation (ensemble average) of the square of the electric field will now be computed:

$$\langle E^2(x,t)\rangle=\int\int d^3k\,d\omega\,d^3k'\,d\omega'\langle E(k,\omega)\cdot E^*(k',\omega')\rangle$$
$$\times\exp\{i[(k-k')\cdot x-(\omega-\omega')t]\}.$$

Thus the expectation of

$$\langle S_\sigma(k,\omega)S^*_{\sigma'}(k',\omega')\rangle$$
$$=\frac{1}{(2\pi)^6}\left\langle\sum_{i,j}\exp(-ik\cdot x_{0i\sigma}+k'\cdot x_{0j\sigma'})\delta(\omega-k\cdot v_{i\sigma})\delta(\omega'-k\cdot v_{j\sigma'})\right\rangle, \tag{159}$$

where σ is a species label, must be computed. In accordance with the Rostoker superposition principle, since the particles have been dressed they may be treated as uncorrelated. Thus for any quantities

$$\langle\alpha_\sigma(x_{i0},v_i)\rangle=\int\int\frac{d^3x_{i0}}{v}d^3v_i f_\sigma(v_i)\sigma_\sigma(x_{i0},v_i)$$

and

$$\langle\alpha_\sigma(x_{i0},v_i)\alpha_{\alpha'}(x_{j0},v_j)\rangle$$
$$=\int\int\int\int\frac{d^3x_{i0}}{V}\frac{d^3x_{j0}}{V}d^3v_i d^3v_j f_\sigma(v_i)f_{\sigma'}(v_j)\alpha_\sigma(i)\alpha_{\sigma'}(j).$$

It quickly follows that

$$\langle S_\sigma(k,\omega)S^*_\sigma(k',\omega')\rangle=\frac{n_0}{(2\pi)^3}\delta(k-k')\delta(\omega-\omega')\delta_{\sigma\sigma'}\int d^3v f_\sigma(v)\delta(\omega-k\cdot v). \tag{160}$$

If now (157) and (160) are inserted in the expression for $\langle E^2(x,t)\rangle$,

$$\langle E^2(x,t)\rangle=\frac{n_0}{(2\pi)^3}\left(\frac{\omega_p^2}{n_0}\right)^2\int\int d^3k\,d\omega\left\{\frac{1}{k_-|\varepsilon_{NL}(k_-,\omega_-)|^2}\right.$$
$$\times\left[|\chi_e(k_-,\omega_-)|^2F_i\left(\frac{\omega_-}{k_-}\right)+|1+\chi_i(k_-,\omega_-)|^2F_e\left(\frac{\omega_-}{k_-}\right)\right]$$
$$\left.+\frac{1}{k_+|\varepsilon_{NL}(k_+,\omega_+)|^2}\left[|\chi_e(k_+,\omega_+)|^2F_i\left(\frac{\omega_+}{k_+}\right)+|1+\chi_i|^2F_e\left(\frac{\omega_+}{k_+}\right)\right]\right\}$$
$$\times\left(\frac{|\hat{k}\times E_0|^2}{|D(k,\omega)|^2}+\frac{|\hat{k}\cdot E_0|^2}{\omega^4|\varepsilon(k,\omega)|^2}\right). \tag{161}$$

Equation (161) is the primary result of this subsection. The intensity in each of the lines is obtained by integrating over each of the possible frequency resonances in turn. Several comments are now in order.

The pump could have been electrostatic as well as electromagnetic; nowhere yet has its electromagnetic property been utilized. Secondly, the formula can be utilized for a sum of pumps as long as their phases are random; thus in a weak turbulence theory the enhanced emission term in the kinetic wave equation can be computed for one $\boldsymbol{k}$ mode due not only to the pump, but also due to the other parametrically excited modes. Several simplifications occur depending upon whether the low-frequency mode is an ion-type mode or an electron-type mode.

For $\omega_\pm \gtrsim k_\pm v_{Te}$, the ion response χ_i and source F_i may be neglected. For $\omega_\pm \lesssim k_\pm v_{Ti}$ the electron source F_e may be neglected and $\chi_e, \chi_i > 1$ may be taken. The formalism is not valid for scattering off electron modes for which $k\lambda_{De} \ll 1$, for then the emission is due to longitudinal bremsstrahlung rather than Cerenkov emission. However, by replacing the source F_e by the appropriate quantity the remainder of the calculation goes through [see Bekefi (1966)].

The special calculation for the electron–ion decay with $T_e \sim T_i$ has been treated in several places in the literature. This is the case of interest in many ionospheric experiments [see Perkins et al. (1974)].

Of interest for laser experiments are also the Brillouin and Raman scattering. Brillouin scattering with $T_e \sim T_i$ will be demonstrated here as a model of the treatment of the resonance. The calculation parallels that of the previous reference.

Consider the case where $D(\boldsymbol{k}, \omega) \approx 0$, whereas $D(\boldsymbol{k} - 2\boldsymbol{k}, \omega - 2\Omega)$, $\varepsilon(\boldsymbol{k}, \omega)$, and $\varepsilon(\boldsymbol{k} - 2\boldsymbol{k}, \omega - 2\Omega)$ are nonresonant. For

$$\omega \cong \Omega \cong \omega_R \equiv \left(\omega_p^2 + k^2c^2\right)^{1/2},$$

$$\varepsilon_{NL}(k_-, \omega_-) D(\boldsymbol{k}, \omega) \cong \varepsilon(k_-, \omega_-)\left[2\Omega\left(\omega_r - \omega - \frac{i\omega_p^2}{2\Omega^2}\nu_{NL}\right)\right], \tag{162}$$

where

$$\nu_{NL} \equiv \nu + k_-^2 |\hat{\boldsymbol{k}} \times \hat{\boldsymbol{v}}_0|^2 \frac{\Omega}{\omega_p^2} \operatorname{Im} \frac{\chi_e(1 + \chi_i)}{\varepsilon(k_-, \omega_-)}. \tag{163}$$

The real part of the nonlinear pump term gives a small frequency shift which is neglected.

If $\omega_R - \Omega \lesssim |\boldsymbol{K}|(T_i/m_i)^{1/2}$ then only the term involving $F_i(\omega_-/k_-) \times |\boldsymbol{k} \times \boldsymbol{E}_0|^2$ contributes in (161).

Integrating over the electromagnetic resonance gives the following result for the normalized electromagnetic field fluctuations intensity in the line around the incident frequency Ω:

$$I_{\boldsymbol{k}, \omega \sim \Omega} \equiv \frac{\langle E^2 \rangle_{\boldsymbol{k}, \omega \sim \Omega}}{4\pi T_e/(2\pi)^3}$$

$$= |\boldsymbol{k} \times \boldsymbol{E}_0|^2 \frac{E_0^2}{4\pi n T_e} \frac{\omega_p}{\Omega} \frac{\Omega k_D}{k_-} \frac{V_{Te}}{v_{Ti}} \left(\frac{\pi}{8}\right)^{1/2} \frac{1}{\nu_{NL}} \times \frac{e^{-x^2}}{|1 + T_e/T_i[1 + Z(x)]|^2}, \tag{164}$$

where $x = (\omega_R - \Omega)/\sqrt{2k}\, v_{Ti}$. Here $Z(x)$ is the plasma dispersion function. The enhancement is identical to that of the electron–ion decay except for the factor ω_p/Ω. The enhancement at other possible resonances can be computed in a similar fashion.

Some remarks on inhomogeneous plasma and higher order effect* $O(\varepsilon^2)$ *in homogeneous plasma

Kent and Taylor (1969) have used the superposition principle to describe fluctuation levels in weakly inhomogeneous plasma, of drift-type waves where convective instability is present but the plasma is globally strongly stable. Baldwin and Callen (1972) have also used this procedure to estimate transport due to loss-cone instabilities (convective) in mirror-type fusion devices. Finally, Nevins and Chen (1980) have calculated fluctuation levels and transport for the collisionless drift wave (which are only weakly globally stable) including the contribution from the global normal modes. This last paper discusses the limitations of these theories.

The $O(\varepsilon)$ theory is not adequate to discuss contributions to the spectral intensity due to bremsstrahlung (longitudinal and transverse) and one must proceed to next order in ε. For thermal equilibrium, the fluctuation-dissipation theorem (108) may be used to "save an order" in the calculation because the temperature T is formally $O(\varepsilon)$. [Remember the plasma limit is $e \to 0$, $m \to 0$, e/m finite, ne finite, v_T finite. Hence $T = \frac{1}{2}mv_T^2$ is $O(\varepsilon)$]. In (108) the quantity $\langle \boldsymbol{EE} \rangle$ is $O(\varepsilon)$ and $\varepsilon(\boldsymbol{k}, \omega)$ is the Vlasov dielectric function. To proceed to collisional order one needs $\varepsilon(\boldsymbol{k}, \omega)$ to next order (or equivalently the conductivity). This procedure has been utilized by Dawson and Oberman (1962), Oberman et al. (1962), Dupree (1963), and Coste (1965), who have calculated the collisional contribution to the conductivity for high-frequency waves, and then related it to the emission. It will merely be pointed out here that the solution of the equation for Δ, and then Γ, in the high-frequency limit is no more complicated than and essentially identical to the solution for the pair correlation function required by the previous authors.

2.3.4. *Kinetic equation for fluctuations on the kinetic scale*

Motivation

This section is concerned with fluctuations with frequency of order $\varepsilon\omega_p$ and wavenumber of order εk_D, that is on the "kinetic" scale.

First recall the multiple-timescale derivation of the kinetic equation for the particle distribution function f. Consider first a stable plasma with no spatial gradients and in the absence of external fields. The one- and two-particle functions f and g satisfy the equations (40) and (41) of Section 2.3.3, namely,

$$\frac{\partial f(x,t)}{\partial t}=\frac{ne^2}{m}\int \mathrm{d}X'\frac{\partial}{\partial x}\frac{1}{|x-x'|}\cdot\frac{\partial}{\partial v}g(X,X',t), \tag{165}$$

$$\left(\frac{\partial}{\partial t}+v\cdot\frac{\partial}{\partial x}+v\cdot\frac{\partial}{\partial x'}\right)g(X,X',t)-\frac{ne^2}{m}\frac{\partial f}{\partial v}\cdot\int \mathrm{d}X''\ \frac{\partial}{\partial x}\frac{1}{|x-x''|}g(X',X'',t)$$
$$-\frac{ne^2}{m}\frac{\partial f}{\partial v'}\cdot\int \mathrm{d}X''\frac{\partial}{\partial x'}\frac{1}{|x'-x''|}g(X,X'',t)$$
$$=\frac{e^2}{m}\frac{\partial}{\partial x}\frac{1}{|x-x'|}\cdot\left(\frac{\partial}{\partial v}-\frac{\partial}{\partial v'}\right)f(X,t)f(X',t). \tag{166}$$

Instead of just making a straightforward perturbation expansion in powers of the plasma parameter ε, which would be invalid for sufficiently long times [of order $(1/\varepsilon\omega_p)$] due to secularities, the *multiple timescale* technique (Frieman, 1963; Sandri, 1963) is used, where one replaces the time variable t by a sequence of formally independent time variables $t, \varepsilon t, \varepsilon^2 t, \ldots$ and the additional freedom introduced is utilized to remove the secularities alluded to earlier. The method is related to the method of averaging of Bogoliubov (1962). f, g, h and $\partial/\partial t$ are thus expanded in the following manner:

$$f=f^{(0)}(t,\varepsilon t,\varepsilon^2 t\ldots x,v)+\varepsilon f^{(1)}(t,\varepsilon t,\varepsilon^2 t\ldots x,v)+\cdots \tag{167}$$
$$g=\varepsilon g^{(1)}(t,\varepsilon t,\varepsilon^2 t\ldots x,v)+\varepsilon^2 g^{(2)}(\ )+\cdots \tag{168}$$
$$h=\varepsilon^2 h^{(2)}(\ )+\cdots \tag{169}$$
$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial \varepsilon t}+\varepsilon^2\frac{\partial}{\partial \varepsilon^2 t}+\cdots \tag{170}$$

Then (165) to zeroth order in ε becomes:

$$\partial f^{(0)}/\partial t=0.$$

The function f is thus a constant on the plasma timescale. To first order (165) and (166) become

$$\frac{\partial f^{(1)}}{\partial t}=-\frac{\partial f^{(0)}}{\partial \varepsilon t}+\frac{ne^2}{m}\int \mathrm{d}X'\frac{\partial}{\partial x}\frac{1}{|x-x'|}\cdot\frac{\partial}{\partial v}g^{(1)}(X,X',t,\varepsilon t\ldots) \tag{171}$$

$$\left(\frac{\partial}{\partial t}+L+L'\right)g^{(1)}=\frac{e^2}{m}\frac{\partial}{\partial x}\frac{1}{|x-x'|}\cdot\left(\frac{\partial}{\partial v}-\frac{\partial}{\partial v'}\right)f^{(0)}(X,\varepsilon t\ldots)f^{(0)}(X',\varepsilon t\ldots), \tag{172}$$

where L is the Vlasov operator,

$$v\cdot\frac{\partial}{\partial x}-\frac{ne^2}{m}\frac{\partial f}{\partial v}\cdot\int \mathrm{d}X''\frac{\partial}{\partial x}\frac{1}{|x-x''|}.$$

In order to ensure that $f^{(1)}$ is nonsecular in t it is required that

$$\frac{\partial f^{(0)}}{\partial \varepsilon t}=\lim_{\substack{t\to\infty\\ \varepsilon t\ \text{finite}}}\frac{ne^2}{m}\int \mathrm{d}X'\frac{\partial}{\partial x}\frac{1}{|x-x'|}\cdot\frac{\partial}{\partial v}g^{(1)}(X,X',t,\varepsilon t,\ldots). \tag{173}$$

Equation (172) may be solved for $g^{(1)}$ obtaining two terms – one from the initial conditions on g which for the most part decay to zero in strongly stable plasma on taking the long time limit, and a second term driven by the ff term (which physically are the correlations created by collisions).

Substituting $g^{(1)}_{t\to\infty}$ into (173) gives the kinetic equation for f, (109), describing its evolution on the collisional timescale. The term arising from the initial correlations can be related to the spectrum of fluctuations present in the plasma. [When the plasma is unstable or only marginally stable this term can dominate and the above procedure fails and one obtains instead the quasilinear equations of Drummond and Pines (1962). Conversely, when the plasma is strongly stable the second term dominates and one obtains the Balescu–Guernsey–Lenard (BGL) equations (110) and (123), which were discussed earlier. Rogister and Oberman (1968, 1969) have derived a comprehensive theory which covers the transition between the Drummond–Pines and BGL theories at marginal stability within the context of weak turbulence theory.]

There are two straightforward extensions of this theory that will be described before the multiple timescale expansion of the fluctuation equations is tackled. The first is the inclusion of spatial gradients and the second is the inclusion of external fields.

The inclusion of spatial gradients in f is straightforward only if the scale of variation is very much longer than a Debye length. This ensures that individual collisions (which take place over distances of about a Debye length, of course) occur in a locally uniform environment. Abandoning this condition would drastically increase the complexity of the collision integral.

One accommodates such spatial gradients and the accompanying electrostatic fields in the theory by using a multiple time- and space-scale expansion, and demanding that the spatial variation in f occurs on the εx scale. Correspondingly, the spatial variation in g is on the $(x-x')$ scale in the difference variable, but on the $\varepsilon(x+x')$ scale in the sum variable. So f and g are expanded as follows:

$$f=f^{(0)}(t,\varepsilon t,\varepsilon^2 t\ldots,\varepsilon x,\varepsilon^2 x\ldots,v)+\varepsilon f^{(1)}(t,\varepsilon t\ldots,\varepsilon x,\varepsilon^2 x\ldots,v)+\mathrm{O}(\varepsilon^2) \tag{174}$$
$$g=\varepsilon g^{(1)}\left[t,\varepsilon t\ldots,x-x',\varepsilon(x-x')\ldots,\varepsilon(x+x'),\varepsilon^2(x+x')\ldots,v,v'\right]+\mathrm{O}(\varepsilon^2). \tag{175}$$

Equations (171) and (172) then become

$$\partial f^{(0)}/\partial t=0$$

in zeroth order, and

$$\frac{\partial f^{(1)}}{\partial t}=\frac{\partial f^{(0)}}{\partial \varepsilon t}-v\cdot\frac{\partial f^{(0)}}{\partial \varepsilon x}-\frac{e}{m}E^0(\varepsilon x,\varepsilon t)\cdot\frac{\partial f^{(0)}}{\partial v}+\int \mathrm{d}X'\frac{\partial}{\partial x}\frac{1}{|x-x'|}\cdot\frac{\partial}{\partial v}g^{(1)}. \tag{176}$$

Equation (172) is unaffected by the new expansion, in lowest order. The effect of inhomogeneity then is to change the left-hand side of (173) to a phase-space derivative; the right-hand side remains unaffected. [Actually spatial inhomogeneity scale need not be $\mathrm{O}(\varepsilon)$ for the argument to hold.]

The other extension of the theory is the inclusion of external fields, which appear on the left-hand sides of both the f and g equations. To obtain the collision integral, one must obtain g by solving (41) with external field terms included. This can be done by the method of integrating along trajectories. In practice one interesting case is that of a uniform (or slowly varying) external magnetic field, which makes the trajectories helices, and the g equation can then be solved by Fourier–Bessel transforms. The point to be made is that the introduction of external fields produces no new problems in principle, but somewhat complicates the results. There are two caveats here. With the self-consistent electric field, the kinetic equation exhibits high-frequency solutions which must be suppressed in order not to invalidate the order in g. This is ameliorated, since $k\lambda_D \ll 1$, by imposing the quasi-neutrality condition $\Sigma e_r n_r(x,t)=0$. The second point is that the presence of, albeit weak, magnetic field gradients gives rise to drifts in velocity and trapping in magnetic wells and these important effects must be incorporated into the expression $v\cdot \partial/\partial x$. These are dealt with by appropriate gyro-averaging (and bounce averaging) to obtain the so-called drift kinetic equation. This is handled in other articles in the series. These details are omitted here so as not to obscure the heuristics of our argument.

It will now be seen that a similar analysis of the following equations for Γ_1 and Δ lead to a kinetic equation for Γ_1. The additional complications arise from the singular initial conditions on Γ_1 and Δ as compared with the smooth initial conditions on f and g.

Recall from (43) and (51) in Section 2.3.2 that Γ_1 and Δ satisfy the following equations and initial conditions:

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+v\cdot\frac{\partial}{\partial x}+\frac{e_r}{m_r c}v\times B_0\cdot\frac{\partial}{\partial v}\right)\Gamma_1(X_0,t_0;X,t)\\&\quad-\frac{e_r}{m_r}\frac{\partial f_r(X,t)}{\partial v}\cdot\sum n_{r'}e_{r'}\int \mathrm{d}X'\,\Gamma_1(X_0,t_0;X',t)\frac{\partial}{\partial x}\frac{1}{|x-x'|}\\&\quad-\frac{e_r}{m_r}\frac{\partial}{\partial v}\Gamma_1(X_0,t_0;X,t)\cdot\sum_{r'}n_{r'}e_{r'}\int \mathrm{d}X'\,f(X',t)\frac{\partial}{\partial x}\frac{1}{|x-x'|}\\&=\frac{e_r}{n_r}\sum_{r'}n_{r'}e_{r'}\int \mathrm{d}X'\,\frac{\partial}{\partial x}\frac{1}{|x-x'|}\cdot\frac{\partial}{\partial v}\Delta(X_0,t_0;X,X',t),\end{aligned}\tag{177}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+v\cdot\frac{\partial}{\partial x}+v'\cdot\frac{\partial}{\partial x'}+\frac{e_r}{m_r c}v\times B_0\cdot\frac{\partial}{\partial v}+\frac{e_{r'}}{m_{r'}c}v'\times B_0\frac{\partial}{\partial v'}\right)\Delta(X_0,t_0;X,X',t)\\&=e_r e_{r'}\frac{\partial}{\partial x}\frac{1}{|x-x'|}\left(\frac{1}{m_r}\frac{\partial}{\partial v}-\frac{1}{m_{r'}}\frac{\partial}{\partial v'}\right)\\&\quad\cdot\left[\Gamma_1(X_0,t_0;X,t)f(x',t)+\Gamma_1(X_0,t_0;X',t)f(X,t)\right]\\&\quad+\sum_{r''}n_{r''}e_{r''}\int \mathrm{d}X''\left(\frac{e_r}{m_r}\frac{\partial}{\partial x}\frac{1}{|x-x''|}\cdot\frac{\partial}{\partial v}\left[f_1(X,t)\Delta(X_0,t_0;X',X'',t)\right.\right.\\&\quad+f_1(X'',t)\Delta(X_0,t_0;X,X',t)+\Gamma_1(X_0,t_0;X'',t)g(X,X',t)\\&\quad\left.\left.+\Gamma_1(X_0,t_0;X,t)g(X',X'',t)\right]+(X\leftrightarrow X')\right),\end{aligned}\tag{178}$$

$$\Gamma_1(X_0,t_0;X,t_0)=\frac{g(X_0,X_1,t_0)}{f(X_0,t_0)}+\frac{1}{n_{r_0}}\delta_{r_1r_0}\delta(X_1-X_0),\tag{179}$$

$$\begin{aligned}\Delta(X_0,t_0;X,X',t_0)&=\frac{h(X_0,X,X',t_0)}{f(X_0,t_0)}\\&\quad+\frac{g(X,X',t_0)}{f(X_0,t_0)}\frac{1}{n_{r_0}}\left[\delta_{r_0r}\delta(X-X_0)+\delta_{r_0r'}\delta(X'-X_0)\right].\end{aligned}\tag{180}$$

The first point to notice is that (177) and (178) are exactly the equations that would be obtained by a formal linearization $f\to f+\Gamma$, $g\to g+\Delta$, of (40) and (41) for f and g as has been pointed out before. This implies that if Γ and Δ were to satisfy the same smoothness conditions (gradient scale lengths, etc.) as f and g, then Γ_1 must satisfy the linearization of the equation satisfied by f.

It will be shown that collisional diffusion acts to smooth the initial singularities in Γ and Δ sufficiently rapidly to justify the multiple scale derivation of a kinetic equation for Γ. All that is required is one simple key estimate. In a time t a test particle will diffuse into a probability ball of radius r given by $r^2\sim Dt$, where $D\sim\nu\lambda^2$ is the diffusion constant. Now $\nu\sim\varepsilon\omega_p$ (actually $l_n\varepsilon\omega_p$ but the Coulomb logarithm ln will be disregarded for estimation purposes) and $\lambda_{mfp}\sim\lambda_D/\varepsilon$. Hence, $r/\lambda_D\sim\varepsilon^{1/2}(\omega_p t)^{1/2}$ and so in a time long compared with a plasma period $1/\omega_p$, but short compared with a collision time $1/\varepsilon\omega_p$, which we will henceforth call an intermediate time, the probability ball will extend over a region much larger than a Debye sphere. (This argument still holds in magnetized plasma only if the gyroradius is much larger than the Debye length, which is assumed hereafter.)

It has thus been argued that Γ_1 satisfies the linearization of the kinetic equations for f. For a stable plasma, which will be our major concern, this equation is the fully linearized BGL equation:

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+v\cdot\frac{\partial}{\partial x}+\frac{eE_0}{m}\cdot\frac{\partial}{\partial v}\right)\Gamma_1^{r_0;r}(X_0,t_0;X,t)=-\sum_s\frac{\partial}{\partial v}\cdot J^{rs},\\&J^{rs}=16\pi^3\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{(e_r^2/m_r)n_s e_s^2 kk}{|\varepsilon(k,k\cdot v)|^2k^4}\\&\quad\cdot\left[\frac{1}{m_s}\left(\frac{\partial f^s}{\partial v'}\Gamma^{r_0r}(X_0,t_0;Xt)+f^r(v)\frac{\partial}{\partial v'}\Gamma^{r_0s}(X_0,t_0;X,t)\right)\right.\\&\quad-\frac{1}{m_r}\left(\frac{\partial f^r}{\partial v}\Gamma^{r_0s}(X_0,t_0;X',t)+f^s(v')\frac{\partial}{\partial v}\Gamma^{r_0r}(X_0,t_0;X',t)\right)\\&\quad\left.-\left(\frac{1}{m_s}\frac{\partial f^s}{\partial v'}f^r(v)-\frac{1}{m_r}\frac{\partial f^r}{\partial v}f^s(v')\right)\left(\frac{\delta\varepsilon}{\varepsilon}+\frac{\delta\varepsilon^*}{\varepsilon^*}\right)\right]\delta(k\cdot v-k\cdot v).\end{aligned}\tag{181}$$

Here

$$\delta\varepsilon\equiv-\sum_r\frac{n_re_r^2}{m_r}\frac{4\pi k}{k^2}\cdot\int\frac{\mathrm{d}^3v}{\omega+k\cdot v+\mathrm{i}\varepsilon}\cdot\frac{\partial}{\partial v}\Gamma_1(X_0,t_0;X,t).$$

There are a number of possible objections to this derivation:

(1) It has been shown that the scale length of the probability ball, Γ_1, is $\lambda_D/\varepsilon^{1/2}$ (justified a posteriori from the kinetic equation). In the multiple space scale analysis

given above for f, it was supposed that the inhomogeneity scale length was $O(\lambda_D/\varepsilon)$, which is longer. However, it has already been argued that all that was required was that the inhomogeneity scale be $\gg \lambda_D$.

Hence, the long-timescale variation of Γ_1 is determined by:

$$\frac{\mathrm{d}\Gamma_1}{\mathrm{d}t} = \frac{ne^2}{m}\int \mathrm{d}X' \frac{\partial}{\partial x}\frac{1}{|x-x'|}\cdot\frac{\partial}{\partial v}\Delta_\infty. \tag{182}$$

which is precisely the linearized kinetic equation.

(2) It has been argued that the singular initial condition on Γ is dissipated by collisional diffusion: what of the initial condition on Δ? For a kinetic equation to exist Δ must relax to a functional of Γ_1 and f, independent of the initial conditions, on the intermediate timescale. It can be argued that strong stability of the plasma is sufficient for this to be the case.

(3) The $\delta\varepsilon$ terms in the collision integral are somewhat unfamiliar looking and one might wonder what significance they have. If f is a local Maxwellian, then the $\delta\varepsilon$ term in the collision integral integrates to zero, and one obtains the usual Fokker–Planck form (Montgomery and Tidman 1964). However, for nonthermal situations which might be turbulent on the kinetic level such terms are present. Thus in the hydrodynamic case the $\delta\varepsilon$ term makes no contribution to the results. Note, however, that even with the $\delta\varepsilon$ terms the collision operator conserves particle numbers, momentum and energy. It is worth noting that the collision operator for the test particle kinetic equation for Ω_1 would *not* be expected to satisfy momentum and energy conservation since, momentum and energy can (and will) be transferred to the field particles.

2.3.5. *Hydrodynamic fluctuations in a plasma*

Introduction

With a kinetic equation at hand, (181), for $\langle\delta f\delta f\rangle$ [using the Klimontovich notation $\langle\delta f(X,t)\delta f(X',t')\rangle$ for $f(X,t)\Gamma_1(X,t;X',t')$] at hand, a theory of hydrodynamic fluctuations in a plasma due to particle discreteness can be derived by considering the hydrodynamic limit, in which length and time scales of interest are taken to be long compared with the mean free path and mean free time, respectively. In a magnetized plasma, the perpendicular wavelength need only be large compared with the Larmor radius.

The results are analogous to those of Landau and Lifshitz (1969) for a one-component fluid, but have wider validity. Landau and Lifshitz showed that hydrodynamic fluctuations of a simple fluid can be described by linearized Navier–Stokes equations driven by random stresses and heat-flow terms, whose time correlations are determined by the fluid viscosity and the thermal conductivity, respectively. Hinton (1970), using a method similar to that to be used here, showed that the Landau–Lifshitz results are valid, not only in thermal equilibrium, but in any nonequilibrium situation where the hydrodynamic equations themselves are valid. The random stresses and heat flows remain the same as in equilibrium, the modification is that the Navier–Stokes equations are linearized about the given nonequilibrium flow, rather than about thermal equilibrium.

Moment equations for the fluctuations

Instead of solving (181) for $\langle\delta f\delta f\rangle$ with initial condition (179) directly, the following equivalent formal procedure is used. The linearized BGL equation for δf is solved, which is written as:

$$\left[\frac{\partial}{\partial t} + v\cdot\frac{\partial}{\partial x} + \frac{e}{m}\left(E + \frac{1}{c}v\times B\right)\cdot\frac{\partial}{\partial v}\right]\delta f^r = \sum_{r'}\left(K^{(1)}_{rr'} + K^{(2)}_{rr'}\right), \tag{183}$$

with initial condition $\delta f \to \delta f(X,t_0)$ as $t\to t_0$, where K has been written for the linearized collision operator of (181). The autocorrelation may then be computed, using (179):

$$\langle\delta f^{r_0}(X_0,t_0)\delta f^r(X,t_0)\rangle = f^{r_0}(X_0,t_0)\Gamma_1^{r_0 r}(X_0,t_0;X,t_0) \tag{184}$$

$$= f^{r_0}(X_0,t_0)\delta_{r_0 r}\delta(X-X_0) + g^{r r_0}(X_0,X,t). \tag{185}$$

Attention is thus directed to (183) which is the linearization of the BGL equation for f^r:

$$\left(\frac{\partial}{\partial t} + v\cdot\frac{\partial}{\partial x} + \frac{e}{m}\left(E^{(0)} + \frac{1}{c}v\times B_0\right)\cdot\frac{\partial}{\partial v}\right)f^r(X,t) = \sum_{r'}\left(C_{rr'}(f^r,f^{r'}\right), \tag{186}$$

where

$$C_{rr'}(f_r\delta f_{r'}) \equiv K^{(1)}_{rr'}, \qquad C_{rr'}(\delta f_r, f_{r'}) \equiv K^{(2)}_{rr'}, \tag{187}$$

$$K_{rr'} \equiv K^{(1)}_{rr'} + K^{(2)}_{rr'}. \tag{188}$$

Certain properties of $C_{rr'}$ carry over to $K_{rr'}$, in particular

$$\int C^{rr'}\mathrm{d}^3v = 0,$$

$$\int m_r v C^{rr'}\mathrm{d}^3v = 0,$$

$$\int \frac{1}{2}m_r v^2 C^{rr'}\mathrm{d}^3v = 0, \tag{189}$$

imply

$$\int K^{rr'}\mathrm{d}^3v = 0,$$

$$\int m_r v K^{rr'}\mathrm{d}^3v = 0,$$

$$\int \frac{1}{2}m_r v^2 K^{rr'}\mathrm{d}^3v = 0. \tag{190}$$

The properties (189) and (190) arise from the fact that all processes (i.e. ionization, recombination) that convert particles from one species to another have been neglected.

Moment equations may be constructed from (183) in the same manner as for (186), that is, by multiplying through by 1, v or $mv^2/2$ and integrating over velocity. It is clear that if linearized hydrodynamic quantities δn_r, $\delta \boldsymbol{u}_r$, δT_r etc., are defined in an appropriate manner, then the equations they satisfy are precisely the linearization of the moment equations for the macroscopic variables n_r, $\boldsymbol{u}_r$, T_r etc. The macroscopic density n_r, velocity $\boldsymbol{u}_r$ and temperature T_r are defined, together with the traceless stress dyadic Π_r and heat flux $\boldsymbol{q}_r$ of the rth species by

$$\int f_r(X,t)\,\mathrm{d}^3v = n_r(x,t), \tag{191}$$

$$\int \boldsymbol{v} f^r(X,t)\,\mathrm{d}^3v = n_r(x,t)\boldsymbol{u}_r(x,t), \tag{192}$$

$$\int \frac{1}{2} m_r \boldsymbol{v}\boldsymbol{v} f^r(X,t_0)\,\mathrm{d}^3v$$
$$= m_r n_r(x,t)\boldsymbol{u}_r(x,t)\boldsymbol{u}_r(x,t) + n_r(x,t)T_r(x,t)\boldsymbol{I} + \Pi_r(x,t), \tag{193}$$

and

$$\frac{1}{2}\int m_r v^2 \boldsymbol{v} f^r(X,t)\,\mathrm{d}^3v = \boldsymbol{q}_r(x,t) + \Pi_r(x,t)\cdot \boldsymbol{u}_r(x,t)$$
$$+ \boldsymbol{u}_r(x,t)\left[\frac{5}{2}n_r(x,t)T_r(x,t) + \frac{1}{2}m_r u_r^2(x,t)\right]. \tag{194}$$

Rewriting (193) and (194) to obtain explicit expressions gives:

$$T_r(x,t) = \frac{1}{n_r(x,t)}\frac{1}{3}\int m_r[\boldsymbol{v} - \boldsymbol{u}_r(x,t)]^2 f^r(X,t)\,\mathrm{d}^3v, \tag{195}$$

$$\Pi_r(x,t) = \int m_r\{[\boldsymbol{v} - \boldsymbol{u}_r(x,t)][\boldsymbol{v} - \boldsymbol{u}_r(x,t)]$$
$$- \tfrac{1}{3}\boldsymbol{I}[\boldsymbol{v} - \boldsymbol{u}_r(x,t)]^2\} f^r(X,t)\,\mathrm{d}^3v, \tag{196}$$

$$\boldsymbol{q}_r(x,t) = \frac{1}{2}\int m_r[\boldsymbol{v} - \boldsymbol{u}_r(x,t)][\boldsymbol{v} - \boldsymbol{u}_r(x,t)]^2 f^r(X,t)\,\mathrm{d}^3v. \tag{197}$$

δn_r, $\delta \boldsymbol{u}_r$, δT_r, $\delta\Pi_r$ and $\delta \boldsymbol{q}_r$ are defined by linearizing (191)–(194), that is

$$\delta n_r(x,t) = \int \delta f^r(X,t)\,\mathrm{d}^3v, \tag{198}$$

$$\delta[n_r(x,t)\boldsymbol{u}_r(x,t)] = \delta n_r(x,t)\boldsymbol{u}_r(x,t) + n_r(x,t)\delta\boldsymbol{u}_r(x,t)$$
$$= \int \boldsymbol{v}\,\delta f^r(X,t)\,\mathrm{d}^3v, \tag{199}$$

$$\delta(\Pi_r + n_r T_r \boldsymbol{I} + m_r n_r \boldsymbol{u}_r \boldsymbol{u}_r) = \int m_r \boldsymbol{v}\boldsymbol{v}\,\delta f^r\,\mathrm{d}^3v, \tag{200}$$

$$\delta\left[\boldsymbol{q}_r + \left(\frac{5}{2}n_r T_r + \frac{1}{2}m_r u_r^2\right)\boldsymbol{u}_r + \Pi_r\cdot\boldsymbol{u}_r\right] = \frac{1}{2}m_r\int \boldsymbol{v}v^2\,\delta f^r(X,t)\,\mathrm{d}^3v. \tag{201}$$

Rewriting explicitly:

$$\delta\boldsymbol{u}_r(x,t) = \frac{1}{n_r(x,t)}\int[\boldsymbol{v} - \boldsymbol{u}_r(x,t)]\,\delta f^r(X,t)\,\mathrm{d}^3v, \tag{202}$$

$$\delta[n_r(x,t)T_r(x,t)] = \frac{m_r}{3}\int[\boldsymbol{v} - \boldsymbol{u}_r(x,t)]^2\,\delta f^r(X,t)\,\mathrm{d}^3v, \tag{203}$$

and

$$\delta\Pi_r(x,t) = m_r\int\Big\{[\boldsymbol{v} - \boldsymbol{u}_r(x,t)][\boldsymbol{v} - \boldsymbol{u}_r(x,t)]$$
$$- \frac{1}{3}\boldsymbol{I}[\boldsymbol{v} - \boldsymbol{u}_r(x,t)]^2\Big\}\,\delta f^r(X,t)\,\mathrm{d}^3v, \tag{204}$$

$$\delta\boldsymbol{q}_r(x,t) = \frac{m_r}{2}\int\left[[\boldsymbol{v} - \boldsymbol{u}_r(x,t)]\left([\boldsymbol{v} - \boldsymbol{u}_r(x,t)]^2 - \frac{5T_r(x,t)}{m_r}\right)\right]\delta f^r(X,t)\,\mathrm{d}^3v$$
$$- \Pi_r(x,t)\cdot\delta\boldsymbol{u}_r(x,t). \tag{205}$$

Note that (205) differs from Hinton's (1970) definition of $\delta\boldsymbol{q}_r$ by the term $\Pi_r\cdot\boldsymbol{u}_r$. However, the above definition of $\delta\boldsymbol{q}_r$ would have appeared more naturally in Hinton's work, as with this redefinition the fluctuating quantities obey the linearized moment equations, and it is the above $\delta\boldsymbol{q}_r$ that is directly related to the fluctuating temperature gradient. Of course, the moment equations are empty shells until $\delta\boldsymbol{q}$ and $\delta\Pi$ are related to the lower moments when the distinction drawn above becomes academic. One might well ask why it was that on taking moments of the linearized Boltzmann equation, Hinton did *not* obtain linearized moments of the Boltzmann equation, yet after closing them by a Chapman–Enskog procedure he did obtain the linearized Navier–Stokes equations? The answer is simply this matter of the definition of $\delta\boldsymbol{q}$.

If further the electron–ion friction force $\boldsymbol{R}$ and the electron ion collisional heat flux Q are defined by:

$$\boldsymbol{R} = \int m_e(\boldsymbol{v} - \boldsymbol{u})C_{ei}\,\mathrm{d}^3v, \tag{206}$$

$$Q = \frac{1}{2}\int m_e(\boldsymbol{v} - \boldsymbol{u})^2 C_{ei}\,\mathrm{d}^3v, \tag{207}$$

and $\delta\boldsymbol{R}$ and δQ by:

$$\delta\boldsymbol{R} = \int m_e(\boldsymbol{v} - \boldsymbol{u})K_{ei}\,\mathrm{d}^3v, \tag{208}$$

$$\delta Q = \frac{1}{2}\int m_e(\boldsymbol{v} - \boldsymbol{u})^2 K_{ei}\,\mathrm{d}^3v, \tag{209}$$

then the moment equations for n, $\boldsymbol{u}$, T may be written as:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i}(nu_i) = 0 \tag{210}$$

which are the continuity equations,

$$mn\left(\frac{\partial}{\partial t}+u_i\frac{\partial}{\partial x_i}\right)u_j=-\frac{\partial}{\partial x_j}nT-\frac{\partial}{\partial x_i}\Pi_{ij}+en\left(E_j+\frac{1}{c}(\boldsymbol{u}\times\boldsymbol{B})_i\right)+R_j \tag{211}$$

are the momentum equation for each species, and

$$\frac{\partial}{\partial t}\left(\frac{nmu^2}{2}+\frac{3}{2}nT\right)+\frac{\partial}{\partial x_i}\left[\left(\frac{5}{2}nT+\frac{1}{2}nmu^2\right)u_i+\Pi_{ji}u_j+q_i\right]$$
$$=enE_iu_i+R_iu_i+Q, \tag{212}$$

are the energy transport equations.

The first term in (212) represents the rate of change of the total energy density of particles of a given species, consisting of the kinetic energy $\frac{1}{2}mu^2$ and the internal energy $3nT/2$. The divergence term represents the total energy flux containing the work done by the total pressure tensor $(p\boldsymbol{I}+\boldsymbol{\Pi})\boldsymbol{\cdot}\boldsymbol{u}$, the macroscopic energy flux $(\frac{3}{2}nT+\frac{1}{2}mnu^2)\boldsymbol{u}$, and the microscopic heat flow $\boldsymbol{q}$. The right-hand side contains the rate of working of the electric field (the magnetic field does no work) $en\boldsymbol{E}\boldsymbol{\cdot}\boldsymbol{u}$, the frictional heating $\boldsymbol{R}\boldsymbol{\cdot}\boldsymbol{u}$, and the heat transfer between species Q. Equations (210)–(212) are the familiar moment equations for a two-species plasma (Braginskii 1965).

With the definitions (198), (202)–(205), (208), and (209) the equations for δn_r, $\delta\boldsymbol{u}_r$ and δT_r are precisely the linearizations of (210), (211), and (212).

The next step is to close these equations by relating Π_r, $\boldsymbol{q}_r$, $\boldsymbol{R}_r$, Q to the lower moments n_r, $\boldsymbol{u}_r$, T_r and to relate $\delta\Pi_r$, $\delta\boldsymbol{q}_r$, $\delta\boldsymbol{R}_r$ and δQ_r to n_r, $\boldsymbol{u}_r$, T_r, δn_r, $\delta\boldsymbol{u}_r$, δT_r, respectively. The basis of this step is the Chapman–Enskog expansion, where one supposes that the length and time scales of interest are long compared with the mean free path and mean free time, respectively. It is precisely this regime that is referred to as hydrodynamic. This process has been carried out in full detail by Braginskii (1965) for the moments of f. Hinton (1970) has shown, for the case of a neutral gas, how the solutions of the Chapman–Enskog equations for the moments of f are related to the solutions of the linearized Chapman–Enskog equations for δf.

For a neutral gas the transport equations are as follows:

$$q_i=-\kappa\,\partial T/\partial x_i, \tag{213}$$

$$\Pi_{ij}=-\eta\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right). \tag{214}$$

The corresponding transport equations for $\delta\boldsymbol{q}$ and $\delta\Pi$ were shown by Hinton to be:

$$\delta q_i=-\kappa\frac{\partial}{\partial x_i}\delta T-\delta\kappa\frac{\partial T}{\partial x_i},\qquad \delta\kappa\equiv\delta T\frac{\partial\kappa}{\partial T}, \tag{215}$$

$$\delta\Pi_{ij}=-\eta\left(\frac{\partial}{\partial x_j}\delta u_i+\frac{\partial}{\partial x_i}\delta u_j-\frac{2}{3}\delta_{ij}\frac{\partial}{\partial x_k}\delta u_k\right)-\delta\eta\Pi_{ij},\qquad \delta\eta\equiv\delta T\frac{\partial\eta}{\partial T}. \tag{216}$$

With the appropriate definition of $\delta\boldsymbol{q}$ noted above (205), equations (215) and (216) are precisely the full linearizations of (213) and (214) which one would intuitively expect. One can readily show that the same procedure may be carried out in the plasma case. There are some complications that although not affecting the argument in any way do muddy the algebra. One additional complication (Braginskii 1965) in the plasma case is that one may expand the collision integrals in powers of the square root of the mass ratio $(m_e/m_i)^{1/2}$, which even for the hydrogen plasma is a small number (1/40). Because of this disparity in masses, the electron and ion components are much more strongly coupled to themselves than to each other and it was meaningful to define separate velocities and temperatures to the two components. In fact,

$$T_{ee}:T_{ii}:T_{ei}=1:(m_e/m_i)^{1/2}:(m_e/m_i),$$

where T_{ee}, T_{ii}, T_{ei}, are respectively the electron, ion, and electron–ion equilibration times. The fact that there are two interpenetrating fluids with different temperatures and velocities introduces new transport coefficients not present in the theory of a simple neutral gas, namely the interspecies friction constant that relates the interspecies friction force $\boldsymbol{R}$ to their relative velocity ($\boldsymbol{u}_e-\boldsymbol{u}_i=\boldsymbol{u}$), and the interspecies thermal transfer coefficient relating the interspecies thermal flux Q to their temperature difference (T_e-T_i).

A further complication (relative to the neutral gas case) is the possible presence of an external DC magnetic field, which makes the (previously scalar) thermal conductivity and viscosity tensor quantities, as transport across the magnetic field is generally inhibited.

These two complications produce a proliferation in the number of transport coefficients, but have no effect on the general result that fluctuating quantities $\delta\boldsymbol{q}$, δQ, $\delta\Pi$, $\delta\boldsymbol{R}$ satisfy the full linearization of the equations satisfied by $\boldsymbol{q}$, Q, Π, and $\boldsymbol{R}$, respectively.

The transport equations satisfied by $\boldsymbol{q}$, Q, Π, and $\boldsymbol{R}$ are as follows (Braginskii 1965). The subscripts $\|$, $\perp$ refer to directions parallel and perpendicular to the external magnetic field, so that:

$$\boldsymbol{u}\equiv\boldsymbol{u}_{\|}+\boldsymbol{u}_{\perp}, \tag{217}$$

$$\nabla x=\nabla_{\|}x+\nabla_{\perp}x, \tag{218}$$

etc.

The transfer of momentum from the ions to the electrons by collisions $\boldsymbol{R}$ is made up of two components, the friction force $\boldsymbol{R}_u$ and the thermal force $\boldsymbol{R}_T$, arising from the relative velocity and electron temperature gradients respectively.

$$\boldsymbol{R}=\boldsymbol{R}_u+\boldsymbol{R}_T, \tag{219}$$

$$\boldsymbol{R}_u=-\alpha_{\|}\boldsymbol{u}_{\|}-\alpha_{\perp}\boldsymbol{u}_{\perp}+\alpha_{\wedge}\hat{\boldsymbol{b}}\times\boldsymbol{u}, \tag{220}$$

$$\boldsymbol{R}_T=-\beta_{\|}^{uT}\nabla_{\|}T_e-\beta_{\perp}^{uT}\nabla_{\perp}T_e-\beta_{\wedge}^{uT}\hat{\boldsymbol{b}}\times\nabla T_e, \tag{221}$$

where $\hat{\boldsymbol{b}}$ is a unit vector parallel to the external magnetic field. Thermal forces due to ion temperature gradients are negligible compared with the electron thermal force because of the mass ratio.

The electron heat flux $\boldsymbol{q}$ similarly consists of two parts:

$$\boldsymbol{q}_e = \boldsymbol{q}_{eu} + \boldsymbol{q}_{eT}, \tag{222}$$

$$\boldsymbol{q}_{eu} = \beta_{\|}^{Tu}\boldsymbol{u}_{\|} + \beta_{\perp}^{Tu}\boldsymbol{u}_{\perp} + \beta_{\wedge}^{Tu}\hat{\boldsymbol{b}} \times \boldsymbol{u}, \tag{223}$$

$$\boldsymbol{q}_{eT} = -\chi_{\|}^{e}\nabla_{\|}T_e - \chi_{\perp}^{e}\nabla_{\perp}T_e - \chi_{\wedge}^{e}\boldsymbol{b} \times \nabla T_e. \tag{224}$$

The thermal force $\boldsymbol{R}_T$ and the Nernst heat flux $\boldsymbol{q}_u$ are related by Onsager's (1931a, b) reciprocity, which implies that

$$T_e\beta_{\|,\perp,\wedge}^{uT} = \beta_{\|,\perp,\wedge}^{Tu}. \tag{225}$$

One should note that this comes from *anti*-symmetry [the signs in (221) and (223) are opposite] of the kinetic coefficients as $\boldsymbol{u}$ is odd under time reversal while ∇T_e is even. The factor of T_e arises because the kinetic coefficients have not been defined in terms of conjugate forces and fluxes; a point that will be made clearer later.

The stress tensor being traceless and symmetric forms a five-dimensional representation of the rotation group. The presence of a magnetic field defines a preferred axis and the viscosity becomes "nondegenerate," so there are five independent viscosities $\eta_{0,1,2,3,4}$:

$$\Pi_{\alpha\beta} = -\eta_0 W_{0\alpha\beta} - \eta_1 W_{1\alpha\beta} - \eta_2 W_{2\alpha\beta} + \eta_3 W_{3\alpha\beta} + \eta_4 W_{4\alpha\beta}. \tag{226}$$

Here

$$\begin{aligned}
W_{0\alpha\beta} &= \tfrac{3}{2}\left(b_\alpha b_\beta - \tfrac{1}{3}\delta_{\alpha\beta}\right)\left(b_\mu b_\nu - \tfrac{1}{3}\delta_{\mu\nu}\right)W_{\mu\nu},\\
W_{1\alpha\beta} &= \left(\delta_{\alpha\mu}^{\perp}\delta_{\beta\nu}^{\perp} + \tfrac{1}{2}\delta_{\alpha\beta}^{\perp}b_\mu b_\nu\right)W_{\mu\nu},\\
W_{2\alpha\beta} &= \left(\delta_{\alpha\beta}^{\perp}b_\beta b_\nu + \delta_{\beta}^{\perp}b_\alpha b_\mu\right)W_{\mu\nu},\\
W_{3\alpha\beta} &= \tfrac{1}{2}\left(\delta_{\alpha\mu}^{\perp}\varepsilon_{\beta\gamma\nu} + \delta_{\beta\nu}^{\perp}\varepsilon_{\alpha\gamma\mu}\right)b_\gamma W_{\mu\nu},\\
W_{4\alpha\beta} &= \left(b_\alpha b_\mu \varepsilon_{\beta\gamma\nu} + b_\beta b_\gamma \varepsilon_{\alpha\gamma\mu}\right)b_\gamma W_{\mu\nu},
\end{aligned} \tag{227}$$

are the projections of $W_{\alpha\beta}$ on the five rotational eigenstates, $(E_{\alpha\beta})$ where

$$\delta_{\perp\alpha\beta} = \delta_{\alpha\beta} - b_\alpha b_\beta, \tag{228}$$

and $\varepsilon_{\alpha\beta\gamma}$ is the alternating tensor. $W_{\alpha\beta}$ is the *rate of strain tensor* defined by:

$$W_{\alpha\beta} = \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} - \frac{2}{3}\delta_{\alpha\beta}\frac{\partial u_\gamma}{\partial x_\gamma}. \tag{229}$$

Finally, $Q_\Delta = \lambda(1/T_i - 1/T_e)$ relates the interspecies heat transfer to their temperature difference.

The species labels (i, e) have been omitted as the above holds for ions and electrons separately. The tensors (226) are written out in detail in Braginskii (1965, p. 252) where expressions for the various transport coefficients in terms of the plasma parameters are also given (pp. 249–253). It should be noted at this point that $\alpha_\wedge$, $\beta_\wedge$, $\chi_\wedge$, η_3, and η_4 are in one sense not genuine transport coefficients as they are not associated with dissipative processes—giving heat flow along isotherms, forces perpendicular to velocity, etc. They arise purely from the gyromotion of the particles around the field lines, the so-called gyrostresses.

We now have a closed set of equations for the moments δn_r, $\delta \boldsymbol{u}_r$, and δT_r of δf_r, namely the linearization of (210)–(212), (220)–(224) and (226), where it is important to note that not only n_r, $\boldsymbol{u}_r$, and T_r are linearized, but also the transport coefficients. These together with their initial correlations form a complete set of equations from which to calculate the thermal correlations of any of these hydrodynamic variables. The initial correlations may readily be computed from (185) by taking moments, where the contribution of the second term (the screening cloud) can be neglected as being of order $(k\lambda_D)^2$ except when computing charge densities where the electron and ion densities cancel to this order.

The Langevin method

Superficially the method prescribed for the calculation of hydrodynamic correlation functions in a plasma is just that of Onsager and Machlup (1953), as described in Landau and Lifshitz (1969), that is, the correlation functions satisfy homogeneous linearized fluid equations with prescribed initial conditions. Indeed, in *thermal equilibrium* they are identical prescriptions. However, the derivation from kinetic theory, rather than from thermodynamics, has shown that the method has a considerably greater range of validity. Namely: (a) that the method is valid whenever the hydrodynamic description itself is valid; and (b) the fluctuations remain small compared with the background quantities. Condition (a) arises in order to validate the Chapman–Enskog procedure and clearly is necessary for the prescription to make any sense. Condition (b) is necessary in order not to violate the ordering assumed in deriving the kinetic equation for the fluctuations. Some comments will be made later on what might be expected were condition (b) to be violated. (The magnetized version of the collision operator has not been used. This is all right when $\omega_p/\Omega_c > 1$. However, in the outer edges of fusion discharges, e.g., the inequality is reversed. Then, the magnetized version of the collision operator should be used. To our knowledge this has not been done.)

In equilibrium fluctuation theory there are two equivalent prescriptions, the Onsager method where one solves homogeneous equations with given initial conditions, and the Langevin method where one computes the steady-state response to driving terms with given correlations. Because the equations are linear, it is clear that such a correspondence must exist.

There are several ways of deriving the Langevin description from our present treatment. One method is to postulate the existence of a Langevin linearized BGL equation for δf, such that δf is the steady-state response (or at any rate varying on the hydrodynamic timescale) to a rapidly fluctuating random source δS.

In the neutral gas case, Bixon and Zwanzig (1969) suggested such a formalism for thermal equilibrium [see also Fox and Uhlenbeck (1970)] and this was generalized to nonequilibrium systems by Hinton (1970).

The autocorrelation of δS can be readily computed in terms of the initial correlation of δf, the result being

$$\langle \delta S(X_0, t_0)\delta S(X, t)\rangle = \delta(t-t_0)\delta(x-x_0)2K_0 f(X_0, t_0)\delta(v-v_0)$$

where K_0 is the linearized (BGL) collision operator. The derivation is entirely analogous to one which will be given shortly in the context of hydrodynamic equations.

The moments of δS can then be related to transport coefficients by relating them to solutions of Chapman–Enskog integral equations. By taking moments of the equations for δf linearized hydrodynamic equations driven by random source terms are obtained (which may be identified as random stress and heat flows) with known correlations. In the neutral gas case (Hinton, 1970) the equations obtained were the same as those of Landau and Lifshitz (1969) except that the hydrodynamic equations were linearized about the flow regime in question, not just thermal equilibrium, and the correlations of the random drivers take the same form in terms of the transport coefficients, but have their space and time dependence in the general case.

The same procedure as Hinton's (1970) may be carried out in the plasma case with the complications mentioned earlier—namely, the possible presence of a DC magnetic field and the decoupling of the ions and electrons by the mass ratio expansion.

Another possible method of derivation of the Langevin hydrodynamic equations is to note that a set of linear equations have been derived for hydrodynamic correlation functions with known initial conditions. Since the equations are linear, the initial value problem may be replaced with a driver problem where the driving term is chosen so as to reproduce the same correlations. This procedure is readily carried through, provided the equations are stable. (This is a necessary condition if the results are to be believed as nonlinear effects must ensue if the correlations grow sufficiently. In the language of fluid mechanics, we must remain in a subcritical regime.)

The simplest method of obtaining the actual results without getting tangled in the details is to use the equilibrium methods of Landau and Lifshitz and generalize them to nonequilibrium. The justification of this approach is in the kinetic methods outlined above. The Landau and Lifshitz prescription is as follows:

Let a_i be the deviations of the thermodynamic variables from their equilibrium values. (In our case, density, electron and ion velocities, and temperatures). For small deviations the relaxation of these quantities to their equilibrium values is given by a linear relations

$$\partial a_i/\partial t = -\lambda_{ij}a_j. \tag{230}$$

The rate of entropy generation within the plasma may also be written in terms of the a_i,

$$\partial S/\partial t = \beta_{ij}\dot{a}_i a_j, \tag{231}$$

where

$$\beta_{ij} = \partial^2 S/\partial a_i \partial a_j. \tag{232}$$

If forces b_i "conjugate" to the fluxes $\dot{a}_i$, are defined by

$$b_i = \beta_{ij}a_j, \tag{233}$$

then the transport relation (230) may be rewritten as

$$\partial a_i/\partial t = -\gamma_{ik}b_k, \tag{234}$$

where

$$\gamma_{ik}\beta_{kl} = \lambda_{ij}, \tag{235}$$

and

$$(\partial S/\partial t) = (\partial a_i/\partial t)b_i. \tag{236}$$

This enables the results of the Langevin theory to be stated in their simplest form. In the Langevin theory the fluctuating quantities a_i are regarded as driven by random forces c_i so that

$$\partial a_i/\partial t = -\gamma_{ik}b_k + c_i. \tag{237}$$

It will now be shown that if the random forces $c_i(t)$ are chosen to have correlations given by

$$\langle c_i(t_0)c_j(t)\rangle = (\gamma_{ij} + \gamma_{ji})\delta(t-t_0) \tag{238}$$

the Langevin equations (237) are equivalent to the Onsager prescription

$$\frac{\partial}{\partial t}\langle a_i(t_0)a_j(t_0)\rangle + \lambda_{jk}\langle a_i(t_0)a_k(t)\rangle = 0, \qquad t > t_0 \tag{239}$$

with the required initial conditions

$$\langle a_i(t_0)a_j(t_0)\rangle = \beta_{ij}^{-1} \tag{240}$$

or equivalently

$$\langle a_i(t_0)b_j(t_0)\rangle = \delta_{ij} \tag{241}$$

by explicitly evaluating the required $\langle c_i c_j\rangle$.

$$\langle c_i(t_0)c_j(t)\rangle = \left(\delta_{ik}\frac{\partial}{\partial t} + \lambda_{ik}\right)\left(\delta_{jl}\frac{\partial}{\partial t} + \lambda_{jl}\right)\langle a_k(t_0)a_l(t)\rangle$$

which

$$= \left(\delta_{ik}\frac{\partial}{\partial t_0} + \lambda_{ik}\right)\left(-\delta_{je}\frac{\partial}{\partial t_0} + \lambda_{je}\right)\langle a_k(t_0)a_e(t)\rangle H(t_0-t)$$

using (239) and microscopic reversibility for sufficiently small $|t-t_0|$, which becomes

$$= \left(\delta_{ik}\frac{\partial}{\partial t_0} + \lambda_{ik}\right)(\delta_{je}\lambda_{km} + \delta_{km}\lambda_{je})\langle a_m(t_0)a_e(t)\rangle H(t_0-t)$$

using (239) with the time arguments exchanged

$$= (\delta_{je}\lambda_{im} + \delta_{im}\lambda_{je})\langle a_m(t_0)a_e(t_0)\rangle\delta(t_0 - t)$$
$$= (\gamma_{ij} + \gamma_{ji})\delta(t - t_0) \tag{242}$$

where the fact that β is symmetric has been used.

This is the key result of the Langevin theory—that the correlations of the random forces are given by the transport coefficients γ_{ij} connecting the fluxes a_i to their conjugate forces b_j.

Onsager (1931) showed that the matrix γ_{ij} is symmetric or antisymmetric depending on the time reversal properties of a_i and a_j, being symmetric if a_i and a_j behave the same under time reversal. This says that random forces that reverse under time reversal are uncorrelated with random forces that do not.

An important point to note is that the relation (239) is invariant under linear transformations of the a_i. This is necessary as the a_i are undefined to this extent. The only requirement was that the a_i were a complete set of thermodynamic variables for the system.

To apply this to the two-species plasma an appropriate set of thermodynamic variables must first be defined. The rate of entropy production is given by (Braginskii 1965)

$$\frac{\partial S}{\partial t} + \nabla\cdot\left(S_e n_e \boldsymbol{u}_e + S_i n_i \boldsymbol{u}_i + \frac{1}{T_e}\boldsymbol{q}_e + \frac{1}{T_i}\boldsymbol{q}_i\right) = \theta_e + \theta_i + \theta_{ei}, \tag{243}$$

$$T_e\theta_e = -\boldsymbol{q}_e\cdot\nabla\ln T_e - \boldsymbol{R}\cdot\boldsymbol{u} - \frac{1}{2}\pi_{e\alpha\beta}W_{e\alpha\beta}, \tag{244}$$

$$T_i\theta_i = -\boldsymbol{q}_i\cdot\nabla\ln T_i - \frac{1}{2}\pi_{i\alpha\beta}W_{i\alpha\beta}, \tag{245}$$

$$\theta_{ei} = Q_\Delta\left(\frac{1}{T_i} - \frac{1}{T_e}\right), \tag{246}$$

where $S = s_e n_e + s_i n_i$ is the plasma entropy density per unit volume, s_e and s_i are the entropy per electron and ion, respectively.

$$s_{e,i} = \frac{3}{2}\ln T_{e,i} - \ln n_{e,i}. \tag{247}$$

Appropriate fluxes are thus $\boldsymbol{q}_e$, $\boldsymbol{q}_i$, $\boldsymbol{R}$, $\Pi_{e\alpha\beta}$, $\Pi_{i\alpha\beta}$, and Q_Δ which are conjugate to the forces

$$\nabla T_e/T_e^2,\ \nabla T_i/T_i^2,\ \frac{1}{2}W_{e\alpha\beta}/T_e,\ \frac{1}{2}W_{i\alpha\beta}/T_i,(1/T_i - 1/T_e),$$

respectively. Equations (219)–(224), (226) relate the fluxes and forces via the transport coefficients. The correlations of the random heat flows and stresses may now be written down using (242) generalized to a continuous system (Landau and Lifshitz, 1969).

$$\langle\delta\boldsymbol{q}_e(\boldsymbol{x},t)\delta\boldsymbol{q}_e(\boldsymbol{x}',t')\rangle = \delta^3(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')2T_e^2\left[\chi_{\parallel e}\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} + \chi_{\perp e}(\boldsymbol{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\right], \tag{248}$$

$$\langle\delta\boldsymbol{q}_i(\boldsymbol{x},t)\delta\boldsymbol{q}_i(\boldsymbol{x}',t') = \delta^3(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')2T_i^2\left[\chi_{\parallel i},\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} + \chi_{\perp i}(\boldsymbol{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\right], \tag{249}$$

$$\langle\delta\boldsymbol{R}(\boldsymbol{x},t)\delta\boldsymbol{R}(\boldsymbol{x}',t')\rangle = \delta^3(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')2T_e\left[\alpha_\parallel\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} + \alpha_\perp(\boldsymbol{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\right], \tag{250}$$

$$\langle\delta\boldsymbol{R}(\boldsymbol{x},t)\delta\boldsymbol{q}_{e,i}(\boldsymbol{x}',t')\rangle = 0, \tag{251}$$

$$\begin{aligned}\langle\delta\Pi^r_{\alpha\beta}(\boldsymbol{x},t)\delta\Pi^{r'}_{\mu\nu}(\boldsymbol{x}',t')\rangle &= 4T_r\delta_{rr'}\delta^3(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')\left[\frac{3}{2}\eta_0\left(b_\alpha b_\beta - \frac{1}{3}\delta_{\alpha\beta}\right)\left(b_\mu b_\nu - \frac{1}{3}\delta_{\mu\nu}\right)\right.\\ &\left. - \eta_1\left(\delta^\perp_{\alpha\mu}\delta^\perp_{\beta\nu} + \frac{1}{6}\delta^\perp_{\alpha\beta}b_\mu b_\nu + \frac{1}{6}\delta^\perp_{\mu\nu}b_\alpha b_\beta\right) - \eta_2\left(\delta^\perp_{\alpha\mu}b_\beta b_\nu + \delta^\perp_{\beta\nu}b_\alpha b_\mu\right)\right].\end{aligned} \tag{252}$$

If $\delta\Pi^r_{\alpha\beta}$ is written in terms of the five independent traceless symmetric tensors, $E_{\alpha\beta}$ in the same way as $W_{\alpha\beta}$ in (227) so that

$$\delta\Pi^r_{\alpha\beta} = \sum_{p=0}^{4}\delta\Pi_p E_{p\alpha\beta}, \tag{253}$$

then

$$\begin{aligned}\langle\delta\Pi^r_p(\boldsymbol{x},t)\delta\Pi^{r'}_q(\boldsymbol{x}',t')\rangle &= 2T_r\delta^3(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')\delta_{pq}\delta_{rr'}\eta^r_p, \qquad p,q \leqslant 2\\ &= 0 \text{ otherwise.}\end{aligned} \tag{254}$$

The $p, q \geqslant 3$ "components" of the stresses are dissipationless as are $\boldsymbol{q}_\wedge$ and $\boldsymbol{R}_\wedge$, the corresponding transport coefficients $\eta_{3,4}, \chi_\wedge, \alpha_\wedge$ therefore do not appear in the correlations above.

Finally the correlation of the random interspecies heat flow is given by:

$$\langle\delta Q_\Delta(\boldsymbol{x},t)\delta Q_\Delta(\boldsymbol{x}',t')\rangle = 2\lambda\delta^3(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'). \tag{255}$$

An apparent difficulty is that the forces ∇T_e, ∇T_i, and $(1/T_e - 1/T_i)$ are not completely independent, in that given ∇T_e, ∇T_i everywhere and T_e and T_i at one point, one may compute $(1/T_e - 1/T_i)$ everywhere. This might lead one to think that there might be some correlation between δQ, $\delta\boldsymbol{q}_e$, and $\delta\boldsymbol{q}_i$. An analysis of the situation reveals that the random heat flows δQ, $\delta\boldsymbol{q}_e$, and $\delta\boldsymbol{q}_i$ are not uniquely definable, but are undetermined to within the following transformation.

$$\begin{aligned}&\delta\boldsymbol{q}_e \to \delta\boldsymbol{q}_e + \delta\boldsymbol{q}_1, \qquad \delta\boldsymbol{q}_i \to \delta\boldsymbol{q}_i + \delta\boldsymbol{q}_2,\\ &\delta Q \to \delta Q - \nabla\cdot\delta\boldsymbol{q}_1 - \nabla\cdot\delta\boldsymbol{q}_2.\end{aligned} \tag{256}$$

This keeps invariant the amount of heat arriving or departing from any small volume of plasma. $\delta\boldsymbol{q}_e$ and $\delta\boldsymbol{q}_i$ can then be chosen so that the correlations are as given in (248), (249), and (255), and is the choice that respects the microscopic mechanisms of heat transfer.

To summarize the results of the Langevin approach: fluctuating hydrodynamic variables satisfy the plasma hydrodynamic equations linearized about the flow regime, driven by random stresses and heat flows whose correlations are known in terms of the transport coefficients of the plasma. These results are plausible because: (1) any perturbation to a given flow regime would be expected to satisfy equations linearized about that regime; (2) the correlations of the driving stresses and heat flows in complete thermal equilibrium are local quantities and are microscopic in origin, and one would thus expect them to be substantially the same in a situation of local thermodynamic equilibrium.

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References

Baldwin, D., and J. Callen, 1972, Phys. Rev. Lett. 28, 1686.
Balescu, R., 1960, Phys. Fluids 3, 52.
Bekefi, G., 1966, Radiation Processes in Plasmas (Wiley, New York).
Bixon, M., and R. Zwanzig, 1969, Phys. Rev. 187, 267.
Bogolubov, N., 1962, Problems of a dynamical theory in statistical mechanics, translation by E.K. Gora, in: Studies in Statistical Mechanics, Vol. 1 (North-Holland, Amsterdam).
Born, M., and H. Green, 1949, A General Kinetic Theory of Liquids (Cambridge University Press, London).
Bowles, K., 1958, Phys. Rev. Lett. 1, 454.
Braginskii, S., 1965, Reviews of Plasma Physics, Ed. M.A. Leontovich (Consultants Bureau, New York).
Callen, H., and R. Greene, 1952, Phys. Rev. 86, 702.
Callen, H. and T. Welton, 1951, Phys. Rev. 83, 834.
Coste, J., 1965, Nucl. Fusion 5, 284.
Dawson, J., and C. Oberman, 1962, Phys. Fluids 5, 517.
Dougherty, J., and D.T. Farley, 1960, Proc. R. Soc. A259, 79.
Drake, J., et al., 1974, Phys. Fluids 17, 778.
Drummond, W., and D. Pines, 1962, Nucl. Fusion Suppl. 3, 1049.
Dupree, T., 1963, Phys. Fluids 6, 1714.
Evans, J., and M. Loewenthal, 1964, Planetary Space Sci. 12, 915.
Fejer, J., 1960, Can. J. Phys. 38, 1114.
Fox, R., and G. Uhlenbeck, 1970, Phys. Fluids 13, 1893 and 2881.
Frieman, E., 1963, J. Math. Phys. 4, 410.
Gordon, W.E., 1958, Proc. IRE 46, 1824.
Green, H., 1952, The Molecular Theory of Fluids (Interscience, New York).
Guernsey, R., 1960, Ph.D. Dissertation, Univ. of Michigan.

Hinton, F., 1970, Phys. Fluids 13, 857.
Jackson, J., 1962, Classical Electrodynamics (Wiley, New York).
Kent, A., and J. Taylor, 1969, Phys. Fluids 12, 209.
Kirkwood J., 1946, J. Chem. Phys. 14, 180.
Kirkwood, J., 1947, J. Chem. Phys. 15, 72.
Klimontovich, Y., 1957, Zh. Eksp. Teor. Fiz. 33, 982 (Engl. transl., 1958, Sov. Phys. JETP 6, 753).
Klimontovich, Y., 1959, Zh. Eksp. Teor. Fiz. 37, 735 (Engl. transl., 1960, Sov. Phys. JETP 10, 524).
Klimontovich, Y., and V. Silin, 1962, Zh. Eksp. Teor. Fiz. 42, 286 [Engl. transl., 1962, Sov. Phys. JETP].
Krommes, J., 1976, Phys. Fluids 19, 649.
Krommes, J., and C. Oberman, 1976, J. Plasma Phys. 16, 193 and 229.
Kubo, R., 1957, J. Phys. Soc. Japan 12, 570 and 1203.
Landau, L., and E. Lifshitz, 1969, *Statistical Physics* (Addison-Wesley, Reading, MA).
Lenard, A., 1960, Ann. Phys. NY 10, 390.
Mayer, J., and M. Mayer, 1940, Statistical Mechanics of Fluids (Wiley, New York).
Montgomery, D., and D. Tidman, 1964, Plasma Kinetic Theory (McGraw-Hill, New York).
Nevins, W., and L. Chen, 1980, Phys. Fluids 23, 10.
Oberman, C., A. Ron, and J. Dawson, 1962, Phys. Fluids 5, 1514.
Oberman, C., and F. Shure, 1963, Phys. Fluids 6, 834.
Onsager, L., 1931a, Phys. Rev. 37, 405.
Onsager, L., 1931b, Phys. Rev. 38, 2265.
Onsager, L., and S. Machlup, 1953, Phys. Rev. 91, 1505.
Perkins, F., C. Oberman, and E. Valeo, 1974, J. Geophys. Res. 79, 1478.
Perkins, F., E. Salpeter, and K. Yngvesson, 1965, Phys. Rev. Lett. 14, 579.
Rogister, A., and C. Oberman, 1968, J. Plasma Phys. 2, 33.
Rogister, A., and C. Oberman, 1969, J. Plasma Phys. 3, 119.
Rosenbluth, M., and N. Rostoker, 1962, Phys. Fluids 5, 776.
Rostoker, N., 1961, Nucl. Fusion 1, 101.
Rostoker, N., 1961, Phys. Fluids 7, 491.
Salpeter, E.E., 1960, Phys. Rev. 120, 1528.
Sandri, G., 1963, Ann. Phys. NY 24, 332.
Sitenko, A., 1967, Electromagnetic Fluctuations in Plasma (Engl. Transl., Academic Press, New York and London).
Valeo, E., C. Oberman, and F. Perkins, 1972, Phys. Rev. Lett. 28, 340.
Williams, E., 1973, Ph.D. Dissertation, Princeton University.
Yvon, J., 1935, La Théorie des Fluides et l'Equation d'Etat; Actualités Scientifiques et Industrielles (Herman et Cie, Paris).